Toposes and Rings

Jonathon Funk
jfunk@qcc.cuny.edu

Queensborough Community College, CUNY

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This is joint work with Simon Henry

Abstract

I shall attempt to explain a part of a broader program of how topos theory and operator algebra theory match. Following the example of what I call a supported C^*-algebra [1], such as a von Neumann algebra, we extend to an arbitrary ring the notions and constructions introduced there. (Familiarity with [1] is not necessary for the purposes of this talk.) I have included an explanation of the Zariski spectrum of a commutative ring in terms of this construction. Ultimately, our goal is to return to C^*-algebras in order to generalize [1] to all C^*-algebras, not just the supported ones.

Discussion: toposes and C^*-algebras

1

What is the common ground shared by toposes and C^*-algebras? How do we match concepts between the two disciplines? Example: how is polar decomposition of an operator interpreted in topos theory?

2

We define a left-cancellative category and a topos of a C^*-algebra, and more generally of a ring, in a manner that to some extent resembles what is done in pseudogroup and inverse semigroup theory [5, 2, 3].

Discussion continued...

3

We work under a certain hypothesis we shall call *a supported C^*-algebra*.

4

The topos interpretation of polar decomposition we shall see is part of a correspondence between quotients of a torsion-free generator of the topos of a C^*-algebra and certain subcategories of its left-cancellative category.

Example: bounded operators on Hilbert space

Support/cosupport

Let $\mathcal{H}$ denote a Hilbert space
Let $B(\mathcal{H})$ denote the C^*-algebra of bounded operators on $\mathcal{H}$.
$\forall\, S, T, R \in B(\mathcal{H})\ \ \mathrm{Ker}(S) \subseteq \mathrm{Ker}(T) \Rightarrow \mathrm{Ker}(SR) \subseteq \mathrm{Ker}(TR)$.

For $T \in B(\mathcal{H})$ let $N(T)$ denote the projection associated with the subspace $\mathrm{Ker}(T)$.
$\forall\, S, T, R\ N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

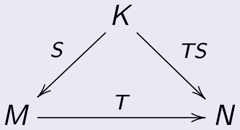
The support projection $C(T) = I - N(T^*)$ is the projection associated with $\overline{\mathrm{Ran}(T)}$.
$\forall\, S, T, R\ :\ C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$

Example continued...

A category associated with $\mathcal{H}$

Let $L(\mathcal{H})$ denote the following category.
Objects: the subspaces of $\mathcal{H}$.
Morphisms: $T : M \longrightarrow N$ is a linear operator T on $\mathcal{H}$ such that $\mathrm{Ker}(T) = M^{\perp}$, and $\mathrm{Ran}(T) \subseteq N$.

$L(\mathcal{H})$ is a category isn't it?



Example continued...

We must have $\mathrm{Ker}(TS) = K^{\perp}$

We have $\mathrm{Ker}(S) = K^{\perp}$ and $\mathrm{Ran}(S) \subseteq M = \mathrm{Ker}(T)^{\perp}$
Therefore, $\mathrm{Ker}(T) \subseteq \mathrm{Ker}(S^*)$

Hence, $\mathrm{Ker}(TS) \subseteq \mathrm{Ker}(S^*S) \overbrace{=}^{\textit{exercise}} \mathrm{Ker}(S)$
The other inclusion $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(TS)$ is trivial.
Therefore, $\mathrm{Ker}(TS) = \mathrm{Ker}(S) = K^{\perp}$

$L(\mathcal{H})$ is left-cancellative

Let $T : M \longrightarrow N$ be a morphism. Let P denote the projection associated with the subspace M: $\mathrm{Ker}(T) = \mathrm{Ker}(P)$.
Suppose that $TS = TR$, where $S, R : K \longrightarrow M$.
Then for any $v \in \mathcal{H}$, we have $S(v) - R(v) \in \mathrm{Ker}(T)$.
Thus, $P(S(v) - R(v)) = 0$, whence
$S(v) = PS(v) = PR(v) = R(v)$. Thus, $S = R$.

Supported C^*-algebra $\mathcal{A}$

Support/cosupport projection

Let $T \in \mathcal{A}$.

- A support projection $C(T)$ satisfies $C(T) \leq P$ iff $T = PT$ (so $T = C(T)T$)
- A cosupport projection $N(T)$ satisfies $P \leq N(T)$ iff $TP = 0$ (so $TN(T) = 0$)

Lemma: If $C(T)$ exists, then $C(TT^*) = C(T)$.

This follows from the C^*-identity $\|TT^*\| = \|T\|^2$

Supported C^*-algebra $\mathcal{A}$ continued

Support hypothesis

We shall say that a C^*-algebra $\mathcal{A}$ *is supported* if:

1. every $T \in \mathcal{A}$ has a support projection $C(T)$ such that
2. $\forall\, S, T, R \,:\; C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$ (Stability).

The support hypothesis has an equivalent cosupport form:

Cosupport

1. every T has a cosupport projection $N(T)$ such that
2. $\forall\, S, T, R \,:\; N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

von Neumann algebra

$B(\mathcal{H})$ and more generally any von Neumann algebra is supported in this sense.

The left-cancellative category $L(\mathcal{A})$

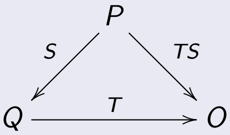
Definition of $L(\mathcal{A})$

Let $\mathcal{A}$ denote a unital supported C^*-algebra.

Objects: projections P of $\mathcal{A}$ ($P^*P = P$)
Morphisms: $T : P \longrightarrow Q$, $C(T^*) = P$ (iff $N(T) = I - P$),
and $T = QT$ ($C(T) \leq Q$)

Another way: a morphism is a pair (T, Q) such that $T = QT$.
Domain of (T, Q) is $C(T^*)$
Codomain of (T, Q) is Q

$L(\mathcal{A})$ is a category



We have $C(S) \leq Q = C(T^*)$.

Then $P = C(S^*) = C(S^*S) \overbrace{\leq}^{\text{stability}} C(S^*T^*) \leq C(S^*)$
Thus, $P = C(S^*T^*) = C((TS)^*)$.
We also have $T = OT$ so of course $TS = OTS$.

The identity morphism $P \longrightarrow P$ is simply P.
Indeed, if $T : P \longrightarrow Q$ is a morphism,
then $TP = TC(T^*) = T$ and $QT = T$.

$L(\mathcal{A})$ is left-cancellative

Suppose that we have morphisms

$$P \underset{R}{\overset{S}{\rightrightarrows}} Q \xrightarrow{T} O \text{ such that } TS = TR\,.$$

Then $T(S-R)=0 \Rightarrow (S^*-R^*)T^*=0$
$\Rightarrow C((S^*-R^*)T^*)=0\,.$
We have $C(Q)=Q=C(T^*)$

Therefore, $C((S^*-R^*)Q) \overbrace{\leq}^{\text{stability}} C((S^*-R^*)T^*)=0$
$\Rightarrow (S^*-R^*)Q=0$
$\Rightarrow Q(S-R)=0 \Rightarrow S=QS=QR=R\,.$

Topos of presheaves on $L(\mathcal{A})$: $\mathscr{B}(\mathcal{A})$

Definition of $\mathscr{B}(\mathcal{A})$

An object of this topos is a functor:
$F : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$

Representable presheaf

Let Q be a projection.
$\widehat{Q} : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$
$\widehat{Q}(P) = L(\mathcal{A})(P, Q) = \{ T \in \mathcal{A} \mid C(T^*) = P \,;\; T = QT \}$
Transition in Q along $S : O \longrightarrow P$: $T \cdot S = TS$ for $C(T^*) = P$

$\mathscr{B}(\mathcal{A})$ continued

Representable presheaf associated with the unit 1

$I = \widehat{1} : L(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathit{Set}$
$I(P) = \{\, T \in \mathcal{A} \mid C(T^*) = P \,\}$
Transition in I along $S : O \longrightarrow P$: $T \cdot S = TS$ for $C(T^*) = P$
(Existence of unit 1 not necessary)

$\mathscr{B}(\mathcal{A})$ is an étendue

The presheaf I is a torsion-free generator [4].

The positive quotient

The presheaf of positive operators

$I^+ : L(\mathcal{A})^{\text{op}} \longrightarrow Set$

$I^+(P) = \{ A \in \mathcal{A} \mid 0 \leq A \, ; \; C(A) = P \}$

Transition in I^+ :

let $S : P \longrightarrow Q$ is a morphism of $L(\mathcal{A})$ and $C(A) = Q$

Define $A \cdot S = S^* A S = (\sqrt{A} S)^* \sqrt{A} S$, which is positive.

Then $C(S^* A S) = C((\sqrt{A} S)^* \sqrt{A} S) = C((\sqrt{A} S)^*) = P$, where

$\sqrt{A} : Q \longrightarrow Q$ is a morphism of $L(\mathcal{A})$; $C(\sqrt{A}) = C(A) = Q$

The quotient map $d : I \longrightarrow I^+$

$d_P : I(P) \longrightarrow I^+(P)$; $d_P(T) = T^* T$

d is a natural transformation: $S^* T^* T S = (TS)^* TS$

d is an epimorphism: if $C(A) = P$, then $d_P(\sqrt{A}) = A$.

Caution: $A \mapsto \sqrt{A}$ is not generally a section of d .

How does this approach generalize to all C^*-algebras?

Is there a C^*-answer?

Should we look for a C^*-algebra or operator algebra approach?
Possibly there is a viable approach using second dual, enveloping von Neumann algebra, and so on.
C^*-expertise needed.

Is there a ring theory and module theory approach?

Yes there is.

Discussion: What is a ring?

Tensor categories or monoidal categories

A ring is a monoid object (category object with a single object) in the tensor category of Abelian groups (with tensor product).

Model theory

A ring is a model of a logical theory.
Examples: C^∞-rings (Dubuc topos); local rings (Zariski topos).

Enrichment

A ring is a monoid enriched in the tensor category of Abelian groups.
Its a preadditive category with a single object.

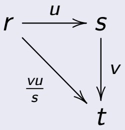
The principal category of a (unital) ring R: $\mathcal{M}(R)$

Objects and morphisms of $\mathcal{M}(R)$ are elements of R

Let $\mathrm{Ann}(r) = \{x \mid rx = 0\}$ = right annihilator of $r \in R$, which is a right ideal.
Objects: elements of R
Morphisms: $u : r \longrightarrow s$ such that:
domain: $\mathrm{Ann}(u) = \mathrm{Ann}(r)$
codomain: $u \leq s$ meaning $u = sq$ or $s \mid u$ or $u \in (s)$ or $(u) \subseteq (s)$
NOTE: an inequality $u \leq s$ is the morphism $u : u \longrightarrow s$ in $\mathcal{M}(R)$.

The principal category ...

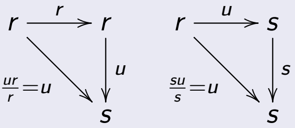
Composition



$$\frac{vu}{s} = vq \; ; \; u = sq \; .$$

If $sq = sp$ then $s(q-p) = 0$ so $v(q-p) = 0$ whence $vq = vp$.

Identity morphism

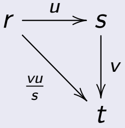


$r = r1$, so $\frac{ur}{r} = u1 = u$. $u = sq$, so $\frac{su}{s} = sq = u$.

The principal category ...

$\mathcal{M}(R)$ is a category

In a triangle below we must have $\mathrm{Ann}(\frac{vu}{s}) = \mathrm{Ann}(r)$, and $\frac{vu}{s} \leq t$.

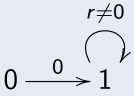


Suppose $u = sq$, so that $\frac{vu}{s} = vq$. So $vq \leq v \leq t$.
We have $\mathrm{Ann}(v) = \mathrm{Ann}(s)$, so that $\mathrm{Ann}(vq) = \mathrm{Ann}(sq)$.
Therefore, $\mathrm{Ann}(\frac{vu}{s}) = \mathrm{Ann}(u) = \mathrm{Ann}(r)$.

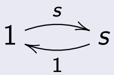
The principal category ...

Example

Let D be an integral domain (commutative, no 0-divisors).
$\mathcal{M}(D)$ is equivalent to this category:



In fact, if $s \neq 0$, then

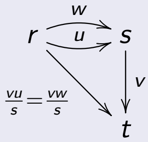


is an isomorphism in $\mathcal{M}(D)$.

The principal category ...

$\mathcal{M}(R)$ is left-cancellative: $\frac{vu}{s} = \frac{vw}{s} \Rightarrow u = w$

Suppose the following diagram commutes.



Suppose $u = sq$, so that $\frac{vu}{s} = vq$, and $w = sp$, so that $\frac{vw}{s} = vp$.
Therefore, $vq = vp$, so that $v(q - p) = 0$.
We have $\mathrm{Ann}(v) = \mathrm{Ann}(s)$, so $s(q - p) = 0$, whence
$u = sq = sp = w$.

The principal category ...

$\mathcal{M}(R)$ is equivalent to the category of (right) principal ideals of R and *injective* R-linear maps between them

Functor: $u : r \longrightarrow s$ to $f_u : (r) \longrightarrow (s)$; $f_u(rp) = up$.
f_u is well-defined: if $rp = rq$, then $r(p - q) = 0$ so $u(p - q) = 0$, whence $up = uq$.
f_u is injective: if $up = f_u(rp) = f_u(rq) = uq$, then $u(p - q) = 0$ so $r(p - q) = 0$, whence $rp = rq$.
This functor is full, faithful, and essentially surjective.
In this sense it is *a weak equivalence*: as far as I can tell there is no functor in the other direction (without choice).

Detour: the Zariski spectrum of a commutative ring

The right divisibility preorder on R

Let DIV denote the preorder

$$u \leq v \text{ if } v \mid u \,, \text{ which means } \exists q \, u = vq \;.$$

This is so iff $u \in (v)$ the principal right ideal generated by v .

The spectral Grothendieck topology on DIV

We shall say that a sieve $S = \{r_\alpha \leq r\}$ is a spectral cover if

$$\exists 0 \leq n \,, \; r^n = \sum_\alpha r_\alpha a_\alpha \;.$$

For instance, if $r = \sum_\alpha r_\alpha a_\alpha$, then S is a spectral cover.

Pullback stability of the spectral topology

Let $S = \{r_\alpha \leq r\}$ be a spectral cover: say $r^n = \sum_\alpha r_\alpha a_\alpha$, $0 \leq n$. Let $s \leq r$, so that $s = rq$. We must show that the pullback sieve

$$s^* S = \{t \leq s \mid t \leq r_\alpha \text{ for some } r_\alpha\}$$

is a spectral cover of s. If $n = 0$, then $s = \sum_\alpha (r_\alpha s) a_\alpha$. If $0 < n$, then

$$s^n = (rq)^n = r^n q^n = \sum_\alpha r_\alpha a_\alpha q^n = \sum_\alpha (r_\alpha q) b_\alpha \,.$$

Also $r_\alpha q = r c_\alpha q = rq c_\alpha = s c_\alpha$, so $r_\alpha q \leq s$.

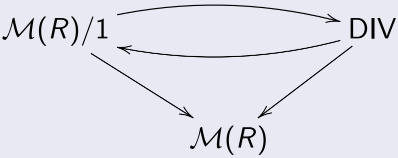
Spectral topology...

Theorem

Let $\mathrm{Spec}(R)$ denote the Zariski spectrum of a commutative ring R . Then the topos of sheaves for the spectral topology on DIV is equivalent to $Sh(\mathrm{Spec}(R))$.

The slice category $\mathcal{M}(R)/1$

$\mathcal{M}(R)/1$ is a preorder. We have an equivalence



Going one way

$$u : r \longrightarrow 1 \ \mapsto u \ ;$$

and back

$$u \mapsto u : u \longrightarrow 1 \ .$$

The principal topos

The annihilator classes

For $r, s \in R$ write $r \sim s$ to mean $\mathrm{Ann}(r) = \mathrm{Ann}(s)$. We say that r and s are in the same annihilator class.
This is a (right) congruence relation $\sim$ on R. Let $\mathrm{Ann} = R/\sim$ denote the set of annihilator classes. Then Ann is a right R-set.
How are the toposes of right R-sets and $\mathscr{M}$ related?

Let $\mathscr{M}(R) = \mathscr{M}$ denote the topos of presheaves on $\mathcal{M}(R)$

$$\text{Yoneda} : \mathcal{M}(R) \longrightarrow \mathscr{M} \; ; \; s \mapsto \widehat{s} \, .$$

$$\widehat{s}(r) = \mathcal{M}(r, s) = \{ u \in R \mid u \sim r \wedge u \leq s \}$$

$$\widehat{1}(r) = \mathcal{M}(r, 1) = \{ u \in R \mid u \sim r \}$$

($\widehat{1}$ is not to be confused with the terminal object of $\mathscr{M}$.)

The principal topos...

Let $A \subseteq R$ be a right R-subset (such as a right ideal)

Define $\widehat{A}(r) = A \cap \widehat{1}(r) = \{\, u \in A \mid u \sim r \,\}$

$$\begin{array}{ccc}
\widehat{A}(r) & \hookrightarrow & \widehat{1}(r) \\
\big\downarrow & & \big\downarrow \\
A & \hookrightarrow & R
\end{array}$$

Then $\widehat{A}$ is a presheaf on $\mathcal{M}(R)$.
Transition in $\widehat{A}$ along $u : r \longrightarrow s$ of $\mathcal{M}(R)$ is given by

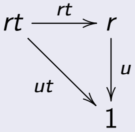
$$v \mapsto \frac{vu}{s} \; ; \; v \in A \text{ . Then } \frac{vu}{s} \in A \text{ .}$$

Moreover, $\widehat{A} \rightarrowtail \widehat{1}$.

The subobject lattice of $\widehat{1}$ in $\mathscr{M}$ and the lattice of right R-subsets of R are isomorphic

Proof

Let $A \subseteq R$ be a right R-subset.
For all $u \in A$, we have $u \in \widehat{A}(u)$.
Suppose $\widehat{A} \rightarrowtail \widehat{B}$ as subobjects of $\widehat{1}$: $\forall r\ \widehat{A}(r) \subseteq \widehat{B}(r)$.
If $u \in A$, then $u \in \widehat{A}(u) \subseteq \widehat{B}(u) \subseteq B$.
In other words, $A \subseteq B$.
Let $S \rightarrowtail \widehat{1}$. Define $A = \bigcup_R S(r)$.
A is a right R-subset of R.
If $u \in S(r)$ then for any $t \in R$ we have $ut \in S(rt)$



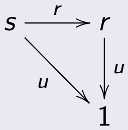
The subobject lattice of $\widehat{1}$...

Proof....

We have $\widehat{A} = S$. In fact, we have

$$\widehat{A}(s) = A \cap \widehat{1}(s) = \bigcup_R S(r) \cap \widehat{1}(s) \subseteq S(s)$$

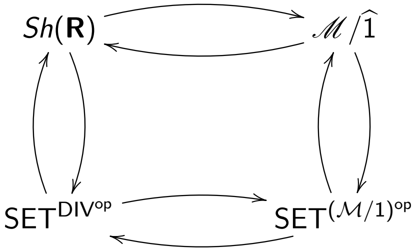
because if $u : r \longrightarrow 1$ is in $S(r)$ and $u \sim s$, then



commutes in $\mathcal{M}(R)$, showing that $u \in S(s)$.
The reverse inclusion is trivial, so that $\widehat{A}(s) = S(s)$,
whence $\widehat{A} = S$.

$\mathscr{M}$ is an étendue

Let **R** denote the topological space with underlying set R topologized by its right R-subsets.
The frame $\mathcal{O}(\mathbf{R})$ and the frame $\mathrm{Sub}(\widehat{1})$ are isomorphic.



Alexandrov: a right mulitiplicative subset of R is the same as a downset of DIV.

The Zariski spectrum Spec(R) ; R commutative

Radical ideals

Let $I \subseteq R$ be an ideal.
Let $\sqrt{I} = \{ r \mid \exists 0 \leq n \, r^n \in I \} \subseteq \bigcap \{ \text{prime } \mathfrak{p} \mid I \subseteq \mathfrak{p} \}$.
Krull: use Zorn's Lemma for $\supseteq$.
Definition: an ideal I is *radical* if $I = \sqrt{I}$.
I is radical iff $\forall n \, (r^n \in I \Rightarrow r \in I)$ iff $I = \bigcap \{ \text{prime } \mathfrak{p} \mid I \subseteq \mathfrak{p} \}$.
The poset of radical ideals ordered by inclusion is a frame denoted

$$\mathcal{O}(\text{Spec}(R))$$

Spec(R)...

A frame morphism

$$
\begin{array}{ccc}
\mathrm{DIV} & & \\
{\scriptstyle r \mapsto \downarrow r = (r)} \downarrow & \searrow^{D} & \\
\mathcal{O}(\mathbf{R}) & \xrightarrow{D^*} & \mathcal{O}(\mathrm{Spec}(R))
\end{array}
$$

$D(r) = \sqrt{r} = \{ s \in R \mid \exists 0 \leq n\,,\ s^n \leq r \} = \bigcap \{ \text{prime } \mathfrak{p} \mid r \in \mathfrak{p} \}$
Eg. $D(1) = R$
Eg. $D(0) = \mathrm{NilRad}(R) = \{ s \in R \mid \exists 0 \leq n\,,\ s^n = 0 \}$

D satisfies:

$D(rs) = D(r) \cap D(s)$ and $D(1) = R$

Spec(R)...

Calculating D^*

The supremum extension is given by

$$D^*(A) = \bigvee_{r\in A} D(r) = \sqrt{\sum_{r\in A} D(r)} = \sqrt{\sum_{r\in A} \sqrt{r}} = \sqrt{\sum_{r\in A} (r)} = \sqrt{A}\,.$$

D is flat

$$D^*(A) \cap D^*(B) = \bigvee_{r\in A} D(r) \cap \bigvee_{s\in B} D(s) = \bigvee_{A\times B} D(r) \cap D(s)$$

$$= \bigvee_{A\times B} D(rs) = \bigvee_{t\in A\cap B} D(t) = D^*(A\cap B)$$

Middle equality. $\leq$: $rs = sr$, so $rs \in A \cap B$
$\geq$: if $t \in A \cap B$, then $(t,t) \in A \times B$ and $D(t^2) = D(t)$.

Spectral topology revisited

The frame morphism $D^* \dashv D_*$ is an inclusion (sublocale)

$D_*(I) = I$, for a radical ideal I.
$D^*(D_*(I)) = D^*(I) = \sqrt{I} = I$.

Geometric morphisms

$$\begin{array}{ccc}
Sh(\mathrm{Spec}(R)) & \xrightarrow{D^* \dashv D_*} & Sh(\mathbf{R}) \simeq \mathscr{M}/\widehat{1} \\
\downarrow & & \downarrow \text{étale surjection} \\
\mathrm{SPEC}(R) & \longrightarrow & \mathscr{M}
\end{array}$$

Spectral topology revisited...

Corollary

The topos $Sh(\mathrm{Spec}(R)$ is a subtopos of $\mathscr{M}/\widehat{1}$ (for the spectral topology on $\mathcal{M}(R)/1$).

Definition

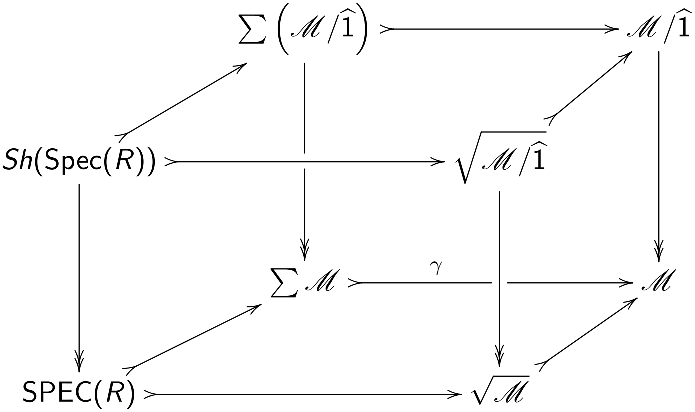
The principal category $\mathcal{M}(R)$ carries a Grothendieck topology, which we call its spectral topology. By definition its subtopos of sheaves is $\mathrm{SPEC}(R)$.

Proposition

The sieve generated by $\{rc_\alpha \leq r\}$ in $\mathcal{M}(R)$ is a spectral cover iff for any $s \sim r$ there is $0 \leq n$ such that

$$s^n = \sum_\alpha sc_\alpha a_\alpha \text{ ; almost all } a_\alpha = 0 \text{ .}$$

Some perSpective



The back face is also a pullback, so that

$$\sum\left(\mathscr{M}/\widehat{1}\right) \simeq \left(\sum\mathscr{M}\right)/\gamma^{*}\widehat{1}\,.$$

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Thank you