

Algebraic-String Diagrams and A Manifest-Covariance Theorem

Ellis D. Cooper

MANIFEST-COVARIANCE-ZS-2.tex, February 16, 2026

email: xtalv1@netropolis.net

Book titles such as "Covariant Physics" (Moataz H. Emam) and "Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory" (Carlo Rovelli and Francesca Vidotto) shout the important of covariance in modern mathematical physics. In categorical terms, covariance is a family of natural isomorphisms of pairs of functors defined on the groupoid of diffeomorphisms in a category of "domains."

"Manifest covariance" is a syntactic concept arising from preservation of covariance of basic covariant tensor calculations combined by composition and product maps.

Differential geometry and general relativity theory calculations are expressible by algebraic string diagrams, including the Einstein Curvature Tensor. Physical nature is categorically natural.

A Practical Introduction to Tensor Networks:
Matrix Product States
and Projected Entangled Pair States

Román Orús

Institute of Physics, Johannes Gutenberg University
55099 Mainz, Germany

June 11, 2014

ADVANCES IN MATHEMATICS 88, 55-112 (1991)

The Geometry of Tensor Calculus, I

André Joyal

Université du Québec à Montréal, Case postale 8888, Succ. "A,"

Montréal. Québec

Canada, H3C3P8

Ross Street

Macquarie University

New South Wales 2109, Australia

Applications of Negative Dimensional Tensors

Roger Penrose

Birkbeck College, University of London, England

$$\text{Diagram 1} + \text{Diagram 2} = \chi_{fde}^b \theta_c^{af} + \chi_{cfe}^a \theta_d^{bf}$$

Philosophical Category Error

Treating a Penrose wire (which denotes an *algebraic identification of indices*) as if it were a String Diagram wire (which denotes a *persistent object undergoing a process*).

Category-theoretic Error

Confusing **contraction** in a tensor expression with **composition** of morphisms in a category. A contraction is a *bilinear operation yielding a scalar or lower-rank tensor*, not a morphism $A \rightarrow B$.

- **Penrose diagrams** are *syntactic devices* for eliminating dummy indices.
- Their semantics is multilinearity and summation, not temporal or causal process.
- **String diagrams** are *semantic representations* of morphism composition.
- Directionality belongs to the *interpretation convention*, not to bare connectivity.

Brooklyn, 1955

- Superman
- Anti-gravity
- “Bobby” Reasenberg

Harvard-Smithsonian Center for Astrophysics Smithsonian Astrophysical Laboratory

R. D. Reasenberg, J. Phillips • Published 1 December 2007 • Physics •
International Journal of Modern Physics D

To test the equivalence principle (EP), we are developing a modern Galilean experiment. In our principle-of-equivalence measurement (POEM), we directly examine the relative motion of two test mass assemblies (TMA) that are freely falling.

In Memoriam
Ralph Abraham 1936-2024

Columbia University, School of Engineering, Spring, 1963		
MASTERPIECES OF THE FINE ARTS	HUMAN C1121Y	D
MASTERPIECES OF MUSIC	HUMAN C1123Y	C+
DIFFERENTIAL GEOMETRY ¹	MATHW3386Y	A-
DIFFERENTIAL GEOMETRY AND RELATIVITY ²	MATH G4512Y	A+

5/17/68

E: Did it baby, it's a Theory.



¹Professor Masatake Kuranishi

²Research Associate Ralph Abraham

Tale

Tensor
Transformation
Theorem

Multi-
linearity
Tensor
Theorem

Christoffel
Transformation
Theorem

Willmore
-Palatini
Theorem

Covariant
Derivative
Theorem

Parallelism
Theorem

Lie Bracket
Theorem

Riemannian
Geometry
Theorem

Manifest
Covariance
Theorem



Take

Tensor
Transformation
Theorem

Multi-
linearity
Tensor
Theorem

Christoffel
Transformation
Theorem

Willmore
-Palatini
Theorem

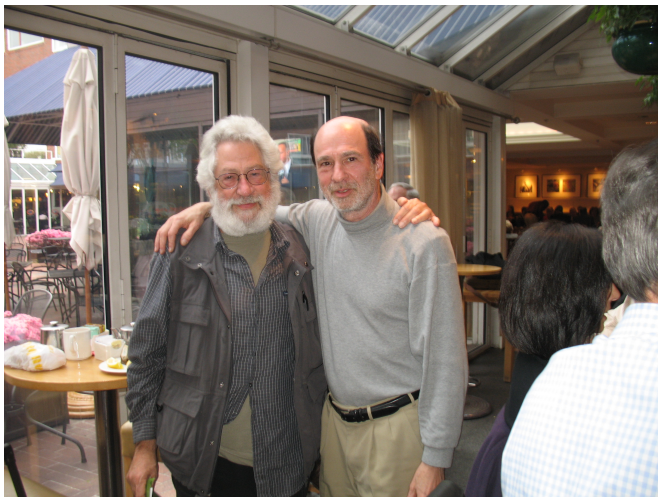
Covariant
Derivative
Theorem

Parallelism
Theorem

Lie Bracket
Theorem

Riemannian
Geometry
Theorem

Manifest
Covariance
Theorem



Example:

A physical theory is said to be covariant with respect to a certain class of transformations when its basic equations retain their 'form' under those transformations. It is one of the basic notions encountered in physics.

“Students’ notions regarding ‘covariance’ of a physical theory,” Atanu Bandyopadhyay and Arvind Kumar, European Journal of Physics, 2010, Volume 31, Number 6

Example:

“We call something invariant, if it does not change under transformations. For instance, let’s consider something arbitrary like $F = F(A, B, C, \dots)$ that depends on different quantities A, B, C, \dots . If we transform $A, B, C, \dots \rightarrow A', B', C', \dots$ and we have

$$F(A', B', C', \dots) = F(A, B, C, \dots)$$

F is called **invariant** under this transformation.” (Jakob Schwichtenberg, “Physics from Symmetry”)

Example:

“As a matter of jargon, we will sometimes refer to quantities written in terms of tensors as covariant (which has nothing to do with “covariant” as opposed to “contravariant”).” Sean Carroll, “Spacetime and Geometry, An Introduction to General Relativity”

Example:

“An equation is called **covariant**, if it takes the same **form** when the objects in it are transformed. For instance, if we have an equation

$$E_1 = aA^2 + bBA + cC^4$$

and after the transformation this equation reads

$$E'_1 = aA'^2 + bB'A' + cC'^4$$

the equation is called **covariant**, because the **form** stayed the same. Another equation

$$E_2 = x^2 + 4axy + z$$

that after a transformation looks like

$$E'_2 = y'^3 + 4az'y' + y'^2 + 8z'x'$$

is not covariant, because it changed its **form** completely.” (Jakob Schwichtenberg, “Physics from Symmetry”)

Category of Domains and Smooth Maps \rightarrow

The objects of $\mathbb{D}\text{om}$ are open sets U of n -fold cartesian products of the real numbers \mathbb{R} , for varying $n \in \mathbb{N}_0$. The morphisms are C^∞ functions $U \xrightarrow{\phi} U'$, $\mathbb{D}\text{om} \subset \text{Set}$, $\mathbb{D}\text{om}(U, V) \subset C^\infty$.

Detail:

- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$;
- $\mathbb{N}_0 = \mathbb{N} - \{0\} = \{1, 2, 3, 4, \dots\}$;
- $n \in \mathbb{N}$, $\text{open } U \subseteq \mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_n$
- **components** (note: $L == R$ means expression L is an *abbreviation* for the expression R .)

$$\begin{aligned}\bar{x}(x) &== \bar{x}(x_1, \dots, x_n) \\ &== (\bar{x}_1(x_1, \dots, x_n), \dots, \bar{x}_m(x_1, \dots, x_n)).\end{aligned}$$

Detail:

- **curve** $(a, b) \subset \mathbb{R}$ (open interval, with 0), $(a, b) \xrightarrow{\gamma} U$
- **scalar field** $U \xrightarrow{f} \mathbb{R}$, $f \in \mathbb{D}_{\text{om}}(U, \mathbb{R}) \subset C^\infty$
- **partial derivative** $U \xrightarrow{\partial_i} \mathbb{R}$, $1 \leq i \leq n$; $\partial_i == \frac{\partial}{\partial x_i} == \frac{\partial}{\partial x_i} |_x$; if $n = 1$, then $\partial_1 == \frac{d}{dt} == \frac{d}{dt} |_{t_0}$;
- **field along curve**: $(a, 0, b) \xrightarrow{\gamma} U \xrightarrow{f} \mathbb{R}$;
- **by an adjunction**, $X \in \mathbb{D}_{\text{om}}(U, \mathbb{R}^n)$ corresponds to $[1, \dots, n] =: [n] \xrightarrow{X} \mathbb{D}_{\text{om}}(U, \mathbb{R})$, and $(X(i))(p) = X^i|_p$.

Category of Finite-Dimensional Vector Spaces

The objects of $\mathbb{F}\mathbb{D}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}$ are finite-dimensional vector spaces $(V, +)$ over the field $(\mathbb{R}, +, \cdot, 0, 1)$ of real numbers.

Details:

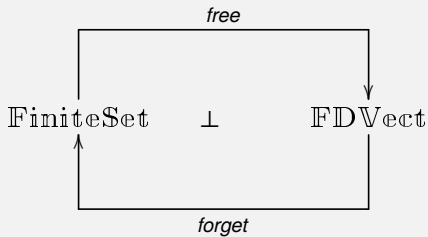
Vector $X \in \mathbb{F}\mathbb{D}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}$ with coordinates $[n] \xrightarrow{X} \mathbb{R}$ has expansion with respect to basis $[b_1, \dots, b_n]$

Alternative Notations:

$$\begin{aligned} X &= X^1 \cdot b_1 + \dots + X^n \cdot b_n \\ &= \sum_{i \in [n]} X^i \cdot b_i \\ &= X^i b_i \\ &= X^i \underbrace{b_i} \end{aligned}$$

Theorem

There is an adjunction



Proof thought:

$$\text{free}\{b_1, \dots, b_n\} := \{X \overset{\square}{\bullet} b \mid X \in \mathbb{R}^n\}$$

$$\text{forget}(V, +) := V$$

Theorem → →

If V is a finite-dimensional vector space, then $V \cong V^{**}$.

Proof →

Define $\varphi : V \rightarrow V^{**}$ by

$$(\varphi(v))(\alpha) = \alpha(v), \quad v \in V, \alpha \in V^*.$$

That is, $\varphi(v)$ is the evaluation functional on V^* at v .

[(i)] φ is injective:

If $\varphi(x) = \varphi(y)$, then for all $\alpha \in V^*$,

$$\alpha(x) = (\varphi(x))(\alpha) = (\varphi(y))(\alpha) = \alpha(y).$$

Hence $\alpha(x - y) = 0$ for all $\alpha \in V^*$, which implies $x - y = 0$, so $x = y$.

Proof

[(ii)] φ is surjective: Choose a basis $(e_j)_{j=1}^n$ of V and its dual basis $(\tilde{e}^i)_{i=1}^n$ of V^* , satisfying

$$\tilde{e}^i(e_j) = \delta_j^i.$$

Let $z \in V^{**}$ and define

$$x := \sum_{i=1}^n z(\tilde{e}^i) e_i \in V.$$

Then for any $\alpha \in V^*$,

$$(\varphi(x))(\alpha) = \alpha(x) = \alpha\left(\sum_i z(\tilde{e}^i) e_i\right) = \sum_i z(\tilde{e}^i) \alpha(e_i) = z\left(\sum_i \alpha(e_i) \tilde{e}^i\right) = z(\alpha),$$

since $\alpha = \sum_i \alpha(e_i) \tilde{e}^i$ in the dual basis. Thus $\varphi(x) = z$, and φ is surjective.

Therefore φ is a linear bijection, hence an isomorphism $V \cong V^{**}$.

Theorem

\mathbf{FDVect} is a symmetric closed rigid monoidal category.

Proof thoughts:

The mathematical structure is $(\mathbf{Vect}, \otimes, \cdot, *, \lambda, \rho, \alpha, \sigma, \text{ev}, \text{coev})$.
Naturally,

$$[\textit{symmetric}] \quad U \otimes V \xLeftrightarrow{\sigma_{U,V}} V \otimes U;$$

[monoidal]

$$\mathbb{R} \otimes V \xLeftrightarrow{\lambda_V} V \xLeftrightarrow{\rho_V} V \otimes \mathbb{R}; \quad (U \otimes V) \otimes W \xLeftrightarrow{\alpha_{U,V,W}} U \otimes (V \otimes W);$$

$$[\textit{closed}] \quad \frac{U \otimes V \rightarrow W}{V \rightarrow W \wedge U}.$$

Proof thought:

[rigidity]

$$\begin{array}{ccc}
 \mathbb{R} \otimes V & \xrightleftharpoons{\lambda_V} & V \xrightleftharpoons{\rho} V \otimes \mathbb{R} \\
 \uparrow \text{ev}_V \otimes 1_V & & \downarrow 1_V \otimes \text{coev}_V \\
 (V \otimes V^*) \otimes V & \xleftarrow{\alpha_{V, V^*, V}} & V \otimes (V^* \otimes V) \\
 & & \xrightarrow{\alpha_{V^*, V, V^*}} (V^* \otimes V) \otimes V^* \\
 \downarrow 1_{V^*} \otimes \text{ev}_V & & \uparrow \text{coev}_V \otimes 1_{V^*} \\
 V^* \otimes \mathbb{R} & \xrightleftharpoons{\rho_V^*} & V^* \xrightleftharpoons{\lambda_{V^*}} \mathbb{R} \otimes V^*
 \end{array}$$

Locally Smooth Maps on a Domain

For $p \in U \in \mathbb{D}om$, $\mathcal{C}U|_p$ denotes the set of C^∞ functions defined in some open neighborhood around p and contained in U .

Tangential-Derivation

For $p \in U \in \mathbb{D}om$, a **tangential-derivation** at p is a linear and Leibnizian function

$$\mathcal{C}U|_p \xrightarrow{\delta} \mathcal{C}U|_p.$$

That means: for a scalar $a \in \mathbb{R}$, $\delta(a \cdot f + g) = a \cdot \delta(f) + \delta(g)$, and $\delta(f \cdot g) = f \cdot \delta(g) + \delta(f) \cdot g$, all defined pointwise.

Theorem

For $p = (x_1, \dots, x_n) \in U \in \mathbb{D}om$ the n functions

$$\mathcal{C}U|_p \xrightarrow{\partial_i|_p} \mathcal{C}U|_p : \partial_i|_p := \frac{\partial}{\partial x_i}|_p, \quad 1 \leq i \leq n$$

are tangential-derivations.

Tangential-Derivation Space

$T_0^1 U|_p$ denotes the set of tangential-derivations at $p \in U$.

Theorem \rightarrow

$(T_0^1 U|_p), +, 0$ is an n -dimensional \mathbb{R} -vector space with basis $[\partial_1|_p, \dots, \partial_n|_p]$ and in which addition $\delta + \delta'$ and scalar multiplication $a \cdot \delta$ are defined pointwise: $(\delta + \delta')f := \delta(f) + \delta'(f)$, $(a \cdot \delta)(f) := a \cdot (\delta(f))$. The basic tangential-derivations $\partial_i|_p$ are called the **coordinate-derivations** at p . There is a canonical isomorphism $T_0^1 U|_p \cong \mathbb{R}^n$ in which $\partial_i|_p$ corresponds to the standard basis vector $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 at the i^{th} position.

Proof references:

Richard L. Bishop and Samuel I. Goldberg, "Tensor Analysis on Manifolds," Theorem 1.7.1, Also see Abraham-Marsden, Theorem 2.2.10, page 83.

Theorem

There are functors

$$\text{Dom} \xrightarrow{T_0^1, T_1^0} \text{FDVect}.$$

Total (Co)Vector Space

The **total tangent space** is the disjoint union of the tangent spaces,

$$T_0^1 U := \dot{\bigcup}_{p \in U} T_0^1 U|_p.$$

Dually, the **total cotangent space** is

$$T_1^0 U := \dot{\bigcup}_{p \in U} T_1^0 U|_p.$$

Definition

The structure

$$T_0^1 U \xrightarrow{\tau_U} U$$

in which $\tau(U)(X_p) := p$ is called the **tangent bundle** over U . A **section** of τ_U is a function $U \xrightarrow{X} T_0^1 U$ such that $\tau_U \circ X = 1_U$. A section X is a **smooth vector field** if the map

$$U \xrightarrow{X} T_0^1 U = \dot{\bigcup}_{p \in U} T_0^1 U|_p \Leftrightarrow \prod_{p \in U} \mathbb{R}^n$$

is in $\text{Dom}(U, \mathbb{R}^n)$.

The set of smooth vector fields over U is denoted by

$$\mathcal{T}_0^1 U \Leftrightarrow \prod_{p \in U} T_0^1 U|_p \quad \text{And dually, } \mathcal{T}_1^0 U \Leftrightarrow \prod_{p \in U} T_1^0 U|_p$$

is the set of smooth covector fields over U .

Theorem ←

A smooth vector field $X \in \mathcal{T}_0^1 U$ determines a linear and Leibnizian function, a **derivation**,

$$\mathcal{T}_0^0 U \xrightarrow{X} \mathcal{T}_0^0 U$$

defined for $f \in \text{Dom}(U, \mathbb{R})$ and $X = X^i \partial_i|_p$ by $X(f)|_p := X^i \partial_i f|_p$.

Jacobian Matrix

If $U \xrightarrow{\phi} \bar{U}$ as in $\bar{x} = \bar{x}(x)$ with $U \subset \mathbb{R}^n$, $\bar{U} \subset \mathbb{R}^m$, then the **Jacobian matrix** of ϕ is (the function $[n] \times [m] \xrightarrow{J\phi} \mathbb{R}$) represented by the array

$$J\phi = \begin{bmatrix} \partial_1 \bar{x}_1 & \cdots & \partial_n \bar{x}_1 \\ \vdots & \partial_i \bar{x}_j & \vdots \\ \partial_1 \bar{x}_m & \cdots & \partial_n \bar{x}_m \end{bmatrix}.$$

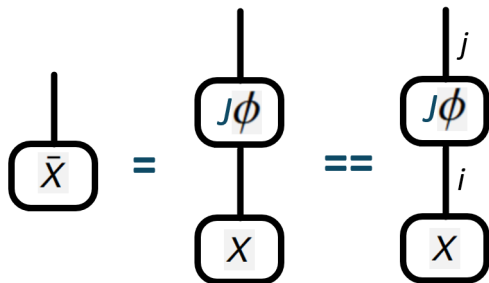
The Jacobian matrix is the coordinate representation of a linear transformation called **pushforward** $\mathcal{T}_0^1 U \xrightarrow{\phi_*} \mathcal{T}_0^1 \bar{U}$ of smooth vector fields:

$$\begin{bmatrix} \bar{X}^1 \\ \vdots \\ \bar{X}^m \end{bmatrix} = \begin{bmatrix} \partial_1 \bar{x}_1 & \cdots & \partial_n \bar{x}_1 \\ \vdots & \partial_i \bar{x}_j & \vdots \\ \partial_1 \bar{x}_m & \cdots & \partial_n \bar{x}_m \end{bmatrix} \begin{bmatrix} X^1 \\ \vdots \\ X^n \end{bmatrix}.$$

in which

$$\bar{X}^j|_{\phi(p)} = \sum_{i \in [n]} X^i \cdot \partial_i \bar{x}_j = \sum_{i \in [n]} X^i \cdot \frac{\partial \bar{x}_j}{\partial x_i} \Big|_p$$

Jacobian Matrix in Algebraic-String Diagram



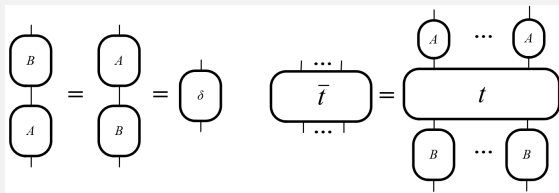
$$\frac{\text{Jacobian}}{\text{Derivative}} \cong \frac{\text{Contraction}}{\text{Integral}}$$

Proposition in Abraham-Marsden

1.7.5 Proposition. Let $\varphi \in L(\mathbf{E}, \mathbf{F})$ be an isomorphism. For ordered bases $\hat{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ of \mathbf{E} and $\hat{\mathbf{f}} = (\mathbf{f}_1, \dots, \mathbf{f}_n)$ of \mathbf{F} , suppose $\varphi(\mathbf{e}_j) = A_j^i \mathbf{f}_i$ and $(\varphi^{-1})^*(\alpha^i) = B_j^i \beta^j$. Then $B_j^i A_k^j = A_j^i B_j^k = \delta_i^k$, or the inverse matrix of (A_j^i) is (B_j^i) , and for $\mathbf{t} \in T_s^r(\mathbf{E})$ with components $t_{j_1 \dots j_s}^{i_1 \dots i_r}$ relative to $\hat{\mathbf{e}}$, the components of $\varphi_s^r \mathbf{t}$ relative to $\hat{\mathbf{f}}$ are given by

$$t_{j'_1 \dots j'_s}^{i'_1 \dots i'_r} = t_{j_1 \dots j_s}^{i_1 \dots i_r} A_{i_1}^{i'_1} \dots A_{i_r}^{i'_r} B_{j'_1}^{j_1} \dots B_{j'_s}^{j_s}$$

Algebraic-String Representation



Theorem ←

There are functors $Mult_S^r$ and T_S^r for which the the values at a finite-dimensional vector space \mathbb{E} are

$$Mult_S^r(\mathbb{E}) := \text{Multilinear}(\underbrace{\mathbb{E}^* \times \cdots \times \mathbb{E}^*}_r \times \underbrace{(\mathbb{E} \times \cdots \times \mathbb{E})}_s, \mathbb{R})$$

$$T_S^r(\mathbb{E}) := \underbrace{(\mathbb{E} \otimes \cdots \otimes \mathbb{E})}_r \otimes \underbrace{(\mathbb{E}^* \otimes \cdots \otimes \mathbb{E}^*)}_s$$

and there is a natural isomorphism

$$\begin{array}{ccc}
 & & Mult_S^r \\
 & & \updownarrow \\
 (FDVect^{op})^r \times FDVect^s & & FDVect \\
 & & \updownarrow \\
 & & T_S^r
 \end{array}$$

Proof without words:

$$\begin{aligned}
 t(X_1, \dots, X_k) &= t(X_1 \lrcorner b, \dots, X_k \lrcorner b) \\
 &= X_1 \lrcorner t(b, X_2 \lrcorner b, \dots, X_k \lrcorner b) \\
 &= X_1 \lrcorner X_2 \lrcorner t(b, \lrcorner b, \dots, X_k \lrcorner b) \\
 &\dots \\
 &= X_1 \lrcorner \dots X_k \lrcorner t(b, \dots, b) \\
 &= X_1 \lrcorner \dots X_k \lrcorner t(b \otimes \dots \otimes b) \\
 &= X_1 \lrcorner \dots X_k \lrcorner t_{\dots}
 \end{aligned}$$

Affine Connection

An **affine connection** on domain U is an \mathbb{R} -bilinear map

$$\mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \xrightarrow{\nabla} \mathcal{T}_0^1 U$$

such that for $f \in \text{Dom}(U, \mathbb{R})$ and $X, Y \in \mathcal{T}_0^1 U$, **derivation**:

$$\nabla_{fX} Y = f \cdot \nabla_X Y;$$

$$\nabla_X fY = f \cdot \nabla_X Y + X(f) \cdot Y.$$

The smooth vector field $\nabla_X Y$ is called the **covariant-derivative** of Y with respect to X . In particular, the covariant-derivative of ∂_j with respect to ∂_i is a smooth vector field, and as such has components as in the expansion

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

Theorem and Proof

Theorem and proof.

$$\begin{aligned}
 \nabla_X Y &= \nabla_{X^i \partial_i} Y^j \partial_j \\
 &= X^i \nabla_i (Y^j \partial_j) \\
 &= X^i (Y^j \nabla_i \partial_j + \partial_j \partial_i Y^j) \\
 &= X^i Y^j \Gamma_{ij}^k \partial_k + X^i Y^j \partial_j \partial_i \\
 &= (X^i Y^j \Gamma_{ij}^k + X(Y^k)) \partial_k \\
 &= (X^i Y^j \Gamma_{ij}^k + X(Y^k)) \partial_k
 \end{aligned}$$

so $(\nabla_X Y)^k = X(Y^k) + Y^j \Gamma_{ij}^k$.

Theorem

For a diffeomorphism $x \rightarrow \bar{x}$ in $\mathbb{D}om$, the transformation law $\Gamma \rightarrow \bar{\Gamma}$ for components of the Christoffel function is

$$\bar{\Gamma}_{ij}^k = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \Gamma_{pq}^r \frac{\partial \bar{x}^k}{\partial x^r} + \frac{\partial \bar{x}^k}{\partial x^p} \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}.$$

Proof

$$\begin{array}{ccc}
 \mathcal{T}_0^1 U|_x \times \mathcal{T}_0^1 U|_x & \xrightarrow{\nabla} & \mathcal{T}_0^1 U|_x \\
 \bar{J} \times \bar{J} \uparrow & \cong & \downarrow J \\
 \mathcal{T}_0^1 \bar{U}|_x \times \mathcal{T}_0^1 \bar{U}|_x & \xrightarrow{\bar{\nabla}} & \mathcal{T}_0^1 \bar{U}|_x
 \end{array}$$

$$\begin{aligned}
 \bar{\Gamma}_{ij}^k \bar{\partial}_k &= \bar{\nabla}(\bar{\partial}_i, \bar{\partial}_j) \\
 &= J(\nabla(\bar{J}\bar{\partial}_i, \bar{J}\bar{\partial}_j)) \\
 &= J(\nabla(\bar{J}_i^p \partial_p, \bar{J}_j^q \partial_q)) \\
 &= J(\bar{J}_i^p (\nabla(\partial_p, \bar{J}_j^q \partial_q))) \\
 &= J(\bar{J}_i^p [\bar{J}_j^q \nabla(\partial_p, \partial_q) + \nabla(\partial_p, \bar{J}_j^q \partial_q)]) \\
 &= J(\bar{J}_i^p [\bar{J}_j^q \Gamma_{\rho q}^r \partial_r] + \bar{J}_i^p [\partial_\rho(\bar{J}_j^q|_{\bar{x}(x)}) \partial_q])
 \end{aligned}$$

Proof, continued

$$\bar{\Gamma}_{ij}^k \bar{J}_k = J(H^r \partial_r) = H^r J(\partial_r) = H^r J_r^k \bar{\partial}_k$$

$$\bar{\Gamma}_{ij}^k = H^r J_r^k$$

$$\bar{\Gamma}_{ij}^k = \bar{J}_i^p \bar{J}_j^q \Gamma_{pq}^r J_r^k + \bar{J}_i^p \left[\partial_p (\bar{J}_j^r) J_r^k \right]$$

Proof, continued

$$\begin{aligned}\bar{J}_i^p \partial_p (\bar{J}_j^r) &= \frac{\partial x_p}{\partial \bar{x}_i} \partial_p \left(\frac{\partial x_r}{\partial \bar{x}_j} \right) \\ &= \frac{\partial x_p}{\partial \bar{x}_i} \frac{\partial}{\partial \bar{x}_b} \frac{\partial x_r}{\partial \bar{x}_j} \frac{\partial \bar{x}_b}{\partial x_p} \\ &= \left(\frac{\partial x_p}{\partial \bar{x}_i} \frac{\partial \bar{x}_b}{\partial x_p} \right) \frac{\partial^2 x_r}{\partial \bar{x}_b \partial \bar{x}_j} \\ &= \left(\delta_b^i \right) \frac{\partial^2 x_r}{\partial \bar{x}_b \partial \bar{x}_j} \\ &= \frac{\partial^2 x_r}{\partial \bar{x}_i \partial \bar{x}_j}\end{aligned}$$

Title

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Proof, concluded.

$$\bar{\Gamma}_{ij}^k = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \Gamma_{pq}^r \frac{\partial \bar{x}^k}{\partial x^r} + \frac{\partial \bar{x}^k}{\partial x^p} \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}$$

$$\begin{array}{c} J^k \\ | \\ \Gamma \\ | \quad | \\ \bar{J}_i \quad \bar{J}_j \end{array} + \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j}$$

Theorem

The covariant-derivative operator extends to the entire tensor algebra.

$$\nabla_{\downarrow} t_J^{\uparrow} = \partial_{\downarrow} t_J^{\uparrow} + \sum_{1 \leq p \leq r} t_J^{\uparrow \hat{i}_p} \left[\Gamma_{\downarrow}^{\uparrow p} \right] - \sum_{1 \leq q \leq s} t_J^{\uparrow} \hat{j}_q \left[\Gamma_{\downarrow}^{\uparrow} \right]_{j_q}$$

Abraham-Marsden, 2.2.17 and p.148 re Willmore's formula. Also, Levi-Civita in "The Absolute Differential Calculus" cites A. Palatini, 1919.

["+" when correcting "up" indices] [interchange and sum on index being corrected]

[differentiating index]

$$S^{\alpha}_{\beta\gamma;\delta} = S^{\alpha}_{\beta\gamma,\delta} + S^{\mu}_{\beta\gamma} \Gamma^{\alpha}_{\mu\delta} - S^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - S^{\alpha}_{\beta\mu} \Gamma^{\mu}_{\gamma\delta}$$

["-" when correcting "down" indices]

[interchange and sum on index being corrected] [differentiating index]

The "rule" as in Misner-Thorne-Wheeler.

Theorem ←

For a diffeomorphism $x \rightarrow \bar{x}$ in $\mathbb{D}om$, the transform of an affine connection is an affine connection. That is to say, if

$$\begin{array}{ccc}
 \mathcal{T}_0^1 U|_x \times \mathcal{T}_0^1 U|_x & \xrightarrow{\nabla} & \mathcal{T}_0^1 U|_x \\
 \bar{J} \times \bar{J} \uparrow & \cong & \downarrow J \\
 \mathcal{T}_0^1 \bar{U}|_{\bar{x}} \times \mathcal{T}_0^1 \bar{U}|_{\bar{x}} & \xrightarrow{\bar{\nabla}} & \mathcal{T}_0^1 \bar{U}|_{\bar{x}}
 \end{array}$$

and ∇ is an affine connection, then $\bar{\nabla}$ is an affine connection.

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Crux of Proof:

$$\begin{aligned}\bar{\nabla}(X, fY) &= (J \circ \nabla \circ (\bar{J} \times \bar{J}))(X, fY) \\ &= (J \circ \nabla)(\bar{J}X, \bar{J}(fY)) \\ &= (J \circ \nabla)(\bar{J}X, f\bar{J}(Y)) \\ &= J(\nabla(\bar{J}X, f\bar{J}(Y))) \\ &= J(f\nabla(\bar{J}X, \bar{J}Y) + (\bar{J}X(f)Y)) \\ &= fJ(\nabla(\bar{J}X, \bar{J}Y)) + J((\bar{J}X)(f)Y) \\ &= fJ(\nabla(\bar{J}X, \bar{J}Y)) + (\bar{J}X)(f)JY .\end{aligned}$$

Covariant-Derivative along a Curve

For a domain U , a curve $(a, 0, b) \xrightarrow{\gamma} U$, and a smooth vector field $U \xrightarrow{X} \mathcal{T}_0^1 U$, the **covariant-derivative of X along γ** is the composition $\nabla_\gamma X := \nabla \circ (J\gamma \circ \pi_{10}, X \circ \pi_{01})$, as in the diagram that proves that covariant-derivative along a curve is covariant:

$$\begin{array}{ccccc}
 \mathcal{T}_0^1(a, b) & \xrightarrow{J\gamma} & \mathcal{T}_0^1 U & & \\
 \pi_{10} \uparrow & & \pi_{10} \uparrow & & \\
 \mathcal{T}_0^1(a, b) \times U & \cdots \cdots \cdots \rightarrow & \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U & \xrightarrow{\nabla} & \mathcal{T}_0^1 U \\
 \pi_{01} \downarrow & & \pi_{01} \downarrow & & \\
 U & \xrightarrow{X} & \mathcal{T}_0^1 U & &
 \end{array}$$

Parallelism (Abraham-Marsden, 2.7.3)

X is **autoparallel along** γ if $\nabla_{\gamma}X = 0$, and γ is a **geodesic** if $J\gamma$ is autoparallel along γ .

Theorem

(i) If $p \in U$ and $V_0 \in \mathcal{T}_0^1 U$, then there is a unique geodesic γ such that $\gamma(0) = p$ and $J\gamma(0) = V_0$. (ii) For any curve $(a, s, t, b) \xrightarrow{\gamma} U$, there is a tangent space isomorphism

$$\mathcal{T}_0^1 U|_{\gamma(s)} \xrightarrow{\tau_{t,s}} \mathcal{T}_0^1 U|_{\gamma t}$$

and functorially, $\tau_{t,s} \circ \tau_{s,r} = \tau_{t,r}$, $\tau_{s,s}$ is the identity map.

Proof note

Solve an ordinary differential equation with two initial conditions.

This is how a connection connects tangent spaces.

Definition

The **Lie-bracket** calculation of smooth vector fields $X, Y \in \mathcal{T}_0^1 U$ is defined by

$$\mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \xrightarrow{[_, _]} \mathcal{T}_0^1 U$$

$$[X, Y]^k \partial_k := (X^i \partial_i) Y^k \partial_k - (Y^j \partial_j) X^k \partial_k$$

Sean Carroll, "Spacetime and Geometry, An Introduction to General Relativity"

Theorem ←

$\mathcal{L} == \mathcal{L}(_, _) := [_, _]$ is covariant.

It must be proved that

$$\overline{\mathcal{L}(X, Y)} = \bar{\mathcal{L}}(\bar{X}, \bar{Y})$$

for any diffeomorphism $\phi \in \mathbb{D}\text{iff}$, in which $\bar{X} = J_\phi(X)$ and $\bar{Y} = J_\phi(Y)$.

Proof, on the left

$$\begin{aligned}
 J_\phi(\mathcal{L}(X^i \partial_i, Y^j \partial_j)) &= J_\phi(((X^i \partial_i) Y^k - (Y^j \partial_j) X^k) \partial_k) \\
 &= ((X^i \partial_i) Y^k - (Y^j \partial_j) X^k) J_k^b \bar{\partial}_b \\
 &= (J_k^b (X^i \partial_i) Y^k - J_k^b (Y^j \partial_j) X^k) \bar{\partial}_b \\
 &= (X^i \frac{\partial \bar{x}_b}{\partial x_k} \frac{\partial Y^k}{\partial x_i} - Y^j \frac{\partial \bar{x}_b}{\partial x_k} \frac{\partial X^k}{\partial x_j}) \bar{\partial}_b \\
 &= (X^i J_k^b \frac{\partial Y^k}{\partial x_i} - Y^j J_k^b \frac{\partial X^k}{\partial x_j}) \bar{\partial}_b .
 \end{aligned}$$

Proof, on the right

$$\begin{aligned}
 \bar{\mathcal{L}}(\bar{X}, \bar{Y}) &= \bar{\mathcal{L}}(\bar{X}^b \bar{\partial}_b, \bar{Y}^c \bar{\partial}_c) \\
 &= (\bar{X}^b \bar{\partial}_b \bar{Y}^l - \bar{Y}^c \bar{\partial}_c \bar{X}^l) \bar{\partial}_l \\
 &= ((J_i^b X^i)(\bar{\partial}_b \bar{Y}^l) - (J_j^c Y^j)(\bar{\partial}_c \bar{X}^l)) \bar{\partial}_l \\
 &= ((J_i^b X^i) \bar{\partial}_b (J_j^l Y^j) - (J_j^c Y^j) \bar{\partial}_c (J_i^l X^i)) \bar{\partial}_l; \text{ and also}
 \end{aligned}$$

$$\bar{\partial}_b (J_j^l \cdot Y^j) = J_j^l \bar{J}_b^j \frac{\partial Y^j}{\partial X_i} + \bar{J}_b^j \frac{\partial^2 \bar{x}_l}{\partial X_i \partial X_j} Y^j$$

$$\bar{\partial}_c (J_i^l \cdot X^i) = J_i^l \bar{J}_c^i \frac{\partial X^i}{\partial X_j} + \bar{J}_c^i \frac{\partial^2 \bar{x}_l}{\partial X_j \partial X_i} X^i \text{ so}$$

$$\begin{aligned}
 &= (J_i^b J_j^l \bar{J}_b^j \frac{\partial Y^j}{\partial X_i} X^i + J_i^b \bar{J}_b^j \frac{\partial^2 \bar{x}_l}{\partial X_i \partial X_j} Y^j X^i \\
 &\quad - J_j^c J_i^l \bar{J}_c^i \frac{\partial X^i}{\partial X_j} Y^j - \bar{J}_c^j \bar{J}_j^c \frac{\partial^2 \bar{x}_l}{\partial X_i \partial X_j} X^i Y^j) \bar{\partial}_l \\
 &= (J_j^l \frac{\partial Y^j}{\partial X_i} X^i - J_i^l \frac{\partial X^i}{\partial X_j} Y^j) \bar{\partial}_l = \overline{\mathcal{L}(X, Y)}.
 \end{aligned}$$

Algebraic-String Diagram

$$\begin{aligned}
 \bar{\mathcal{L}}(\bar{X}, \bar{Y}) &= \bar{\mathcal{L}}(\bar{X} \bar{\partial}, \bar{Y} \bar{\partial}) \\
 &= (\bar{X} \bar{\partial} \bar{Y} - \bar{Y} \bar{\partial} \bar{X}) \bar{\partial} \\
 &= ((J \cdot X)(\bar{\partial} \bar{Y}) - (J \cdot Y)(\bar{\partial} \bar{X})) \bar{\partial} \\
 &= ((J \cdot X) \bar{\partial} (J \cdot Y) - (J \cdot Y) \bar{\partial} (J \cdot X)) \bar{\partial}, \text{ and also} \\
 \bar{\partial} (J \cdot Y) &= J \bar{J} \frac{\partial Y}{\partial x} + \bar{J} \frac{\partial^2 \bar{x}}{\partial x \partial x} Y \\
 \bar{\partial} (J \cdot X) &= J \bar{J} \frac{\partial X}{\partial x} + \bar{J} \frac{\partial^2 \bar{x}}{\partial x \partial x} X \text{ so} \\
 &= (J \bar{J} \frac{\partial Y}{\partial x} X) + (J \bar{J} \frac{\partial^2 \bar{x}}{\partial x \partial x} Y X) \\
 &\quad - J \bar{J} \frac{\partial X}{\partial x} Y - (J \bar{J} \frac{\partial^2 \bar{x}}{\partial x \partial x} X Y) \bar{\partial} \\
 &= (J \frac{\partial Y}{\partial x} X) - J \frac{\partial X}{\partial x} Y \bar{\partial} = \overline{\mathcal{L}(X, Y)}.
 \end{aligned}$$

Geometry of notation for geometry?

Definition: Domain Groupoid

For $n \in \mathbb{N}$, define $\mathbb{D}\text{iff}$ to be the category in which:

- objects are n -dimensional domains $U \in \mathbb{D}\text{om}$;
- morphisms are diffeomorphisms $f : U \rightarrow \bar{U}$.

All morphisms are invertible, so $\mathbb{D}\text{iff}$ is a groupoid,

$$\mathbb{D}\text{iff} \subset \mathbb{D}\text{om} \subset \text{Set}.$$

**Change of coordinates is an *action*
of a groupoid, not a formula.**

Geometric Object

A **geometric object** is a functor

$$\mathcal{F} : \text{Diff} \longrightarrow \text{FDVect},$$

assigning:

- a finite-dimensional vector space $\mathcal{F}(U)$ to each domain U ;
- a linear isomorphism $\mathcal{F}U \xrightarrow{\mathcal{F}\phi} \mathcal{F}\bar{U}$ to each diffeomorphism $U \xrightarrow{\phi} \bar{U}$.

**A geometric object is a
representation of the diffeomorphism
groupoid.**

Tensorial Geometric Object

For $0 < n \in \mathbb{N}, r, s \in \mathbb{N}$, define the **(r,s)-index**

$$\mathbb{I}_s^r := \underbrace{[n] \times \cdots \times [n]}_r \times \underbrace{[n]^T \times \cdots \times [n]^T}_s$$

with typical element

$$((i_1, \dots, i_r), (j_1, \dots, j_s)).$$

Since $\text{Dom}(U, \mathbb{R})$ is an (infinite-dimensional) \mathbb{R} -vector space, so is $\text{Dom}(U, \mathbb{R})^{\wedge} \mathbb{I}_s^r$, which is the vector space of (r,s)-indexed smooth scalar fields on domain U . (Also, they are modules over the unital commutative ring of smooth scalar fields.)

Tensorial Geometric Object, continued

For domain $U \subseteq \mathbb{R}^n$, define

$$\mathcal{T}_s^r(U) := \text{Dom}(U, \mathbb{R})^{\wedge r} \mathbb{I}_s^r,$$

the vector space of (r, s) -indexed smooth scalar fields

$$U \xrightarrow{t} \mathbb{R}^{\wedge r} \mathbb{I}_s^r.$$

If $\phi : U \rightarrow \bar{U}$ is a diffeomorphism, with Jacobian

$$J_\phi(x) \in \text{GL}(n, \mathbb{R}), \quad J_{\phi^{-1}}(\bar{x}) \in \text{GL}(n, \mathbb{R})$$

define

$$\mathcal{T}_s^r(U) \xrightarrow{\mathcal{T}_s^r(\phi)} \mathcal{T}_s^r(\bar{U}) \leftarrow.$$

Definition: Covariant Calculation

Let $\mathcal{F}_1, \dots, \mathcal{F}_k, \mathcal{G}$ be geometric-object functors.

A **covariant calculation** is a multi-linear natural transformation

$$\mathcal{F}_1 U \times \dots \times \mathcal{F}_k U \xrightarrow{\eta} \mathcal{G}.$$

That is, for a diffeomorphism $f : U \rightarrow \bar{U}$, there is a naturality diagram

$$\begin{array}{ccc}
 \mathcal{F}_1 U \times \dots \times \mathcal{F}_k U & \xrightarrow{\eta_U} & \mathcal{G}U \\
 \mathcal{F}_1 \phi \times \dots \times \mathcal{F}_k \phi \downarrow & \parallel & \downarrow \mathcal{G}\phi \\
 \mathcal{F}_1 \bar{U} \times \dots \times \mathcal{F}_k \bar{U} & \xrightarrow{\eta_{\bar{U}}} & \mathcal{G}\bar{U}
 \end{array}$$

Covariance = Naturality

Curvature

Defn. Curvature is a tri-linear map

$$T_0^1 \times T_0^1 \times T_0^1 \xrightarrow{R} T_0^1$$

defined by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

On bases, $R_{ijk}^l :=$

$$R(\partial_i, \partial_j, \partial_k)^l = (\nabla_i \nabla_j \partial_k)^l - (\nabla_j \nabla_i \partial_k)^l$$

since $[\partial_i, \partial_j] = 0$.

Curvature

$$\begin{aligned}(\nabla_i (\nabla_j \partial_k))^l &= (\nabla_i (\Gamma_{jk}^a \partial_a))^l \\&= (\Gamma_{jk}^a \nabla_i \partial_a + \partial_a \nabla_i \Gamma_{jk}^a)^l \\&= (\Gamma_{jk}^a \nabla_i \partial_a + \partial_a \partial_i \Gamma_{jk}^a)^l \\&= (\Gamma_{jk}^a \Gamma_{ia}^l \partial_l + \partial_i \Gamma_{jk}^l \partial_l)^l \\&= (\Gamma_{jk}^a \Gamma_{ia}^l + \partial_i \Gamma_{jk}^l) \partial_l \\ \therefore (\nabla_i \nabla_j \partial_k)^l &= \Gamma_{jk}^l \Gamma_{ij}^l + \partial_i \Gamma_{jk}^l \\&= \partial_i \Gamma_{jk}^l + \Gamma_{jk}^l \Gamma_{ij}^l\end{aligned}$$

$$\begin{aligned}(X^i \partial_i)^i &= X^i \\(X^b \partial_b)^i &= X^i\end{aligned}$$

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Tensor Transformation Theorem

Multi-linearity Tensor Theorem

Christoffel Transformation Theorem

Willmore-Palatini Theorem

Covariant Derivative Theorem

Parallelism Theorem

Lie Bracket Theorem

Riemannian Geometry Theorem

Manifest Covariance Theorem

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Curvature

Since $(\nabla_i \nabla_j \alpha_k)^l = \partial_i \Gamma_{jk}^l + \Gamma_{jk}^l \Gamma_{ij}^l$,
interchanging i, j yields

$$R_{ijk}^l = (\partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l) + (\Gamma_{jk}^l \Gamma_{ij}^l - \Gamma_{ik}^l \Gamma_{jj}^l)$$

or,

$$R_{ijk}^l = (\partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l) + (\Gamma_{jk}^l \Gamma_{ij}^l - \Gamma_{ik}^l \Gamma_{jj}^l)$$

Lemma

The assignment $U \mapsto \mathcal{T}U, f \mapsto Jf$ defines a functor $\text{Dom} \xrightarrow{\mathcal{T}} \text{Alg}$; that is, for $U \xrightarrow{f} V \xrightarrow{g} W$ we have

$$\mathcal{T}(g \circ f) = \mathcal{T}g \circ \mathcal{T}f, \quad \mathcal{T}(\text{id}_U) = \text{id}_{\mathcal{T}U}.$$

At the conceptual level, the pushforward $f_* == \mathcal{T}f$ is the linear map whose coordinate representation is the Jacobian Jf .

Riemannian Metric

A symmetric positive-definite bilinear map

$$\mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \xrightarrow{g} \mathbb{R}$$

is called a **Riemannian metric** for domain $U \in \text{Dom}$. Since there is a canonical bijection $\mathcal{T}_0^1 U|_\rho \cong \mathbb{R}^n$ for ρ in domain U , each tangent space inherits the standard scalar product (a contraction) from \mathbb{R}^n . **Hence, every domain has a Riemannian metric.** (Regarding existence of a Riemannian metric for a differentiable manifold, see Abraham-Marsden 2.5.13, re “partition of unity.”)

Musical Isomorphism Proposition

$$g \in T \iff T_2^0$$

definition

$$T \times T \xrightarrow{g} \mathbb{R}$$

closed

$$T \xrightarrow{g_b} \mathbb{R} \wedge T =: T^*$$

g positive definite

$$g_b \text{ is injective}$$

T is finite-dimensional

$$g_b \text{ is invertible}$$

$$T^* \xrightarrow{g^\# := (g_b)^{-1}} T$$

definition ←

$$T^* \xrightarrow{g^\#} T^{**}$$

closed

$$T^* \times T^* \xrightarrow{\bar{g}} \mathbb{R}$$

$$\bar{g} \in T_0^2$$

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Musical Lemma

If g is a Riemannian metric. then for any diffeomorphism $f : U \rightarrow \bar{U}$, the index-lowering and index-raising maps

$$b : \mathcal{T}_0^1 U \rightarrow \mathcal{T}_1^0 \bar{U}, \quad \sharp : \mathcal{T}_1^0 U \rightarrow \mathcal{T}_0^1 \bar{U}$$

are covariant:

$$f_*(bX) = b(f_*X), \quad f_*(\sharp\alpha) = \sharp(f_*\alpha).$$

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Fundamental Theorem of Riemannian Geometry in Dom

If $\nabla g = 0$ (“metric compatibility”) and $\Gamma_{ij}^k = \Gamma_{ji}^k$ (“torsion free”) then Γ is determined by g (“Levi-Civita connection”).

Theorem

For Riemannian metric $g \in \mathcal{T}_2^0 U$ there exists an affine connection $\nabla(g)$ for which the Christoffel $\Gamma = \Gamma(g)$ coefficients are defined by

$$\Gamma_{||}^{|} = \frac{1}{2} g^{||} (\partial_1 g_{||} + \partial_1 g_{||} - \partial_1 g_{||}) .$$

Proof, Part 1

The diagram illustrates the proof of the Christoffel symbol definition using tensor notation. It shows three rows of equations, each representing a component of the metric compatibility condition $\nabla g = 0$. The first row shows the full equation: $\nabla g = \partial g - \Gamma g - g \Gamma = 0$. The second row shows the same equation with the Γg term crossed out and the $g \Gamma$ term highlighted with a green checkmark. The third row shows the equation with the Γg term highlighted with a green checkmark and the $g \Gamma$ term crossed out. The final result is shown as $0 = \partial g - \Gamma g - g \Gamma$, where the Γg term is crossed out and the $g \Gamma$ term is highlighted with a green checkmark.

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Proof, Part 2

$$0 = \partial_{\mu} g_{\nu\rho} - \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} + 2\Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho}$$

Basic Proposition

Over a domain U and if $0 < k, r, s, r', s', 1 \leq p \leq r, 1 \leq q \leq s$ in \mathbb{N} , then the following basic tensor calculations are covariant:

contraction	$\mathcal{T}_s^r \xrightarrow{- _q^p} \mathcal{T}_{s-1}^{r-1}$
addition	$\mathcal{T}_s^r \times \mathcal{T}_s^r \xrightarrow{-+ -} \mathcal{T}_s^r$
scalar field multiplication	$\mathcal{T}_0^0 \times \mathcal{T}_s^r \xrightarrow{-\cdot -} \mathcal{T}_s^r$
outer product	$\mathcal{T}_s^r \times \mathcal{T}_{s'}^{r'} \xrightarrow{-\otimes -} \mathcal{T}_{s+s'}^{r+r'}$
index-raise	$\mathcal{T}_s^r \xrightarrow{b^k -} \mathcal{T}_{s-1}^{r+1}$
index-lower	$\mathcal{T}_s^r \xrightarrow{\#_k -} \mathcal{T}_{s+1}^{r-1}$
Lie-bracket	$\mathcal{T}_0^1 \times \mathcal{T}_0^1 \xrightarrow{[-, -]} \mathcal{T}_0^1 \rightarrow$
covariant derivative	$\mathcal{T}_0^1 \times \mathcal{T}_0^1 \xrightarrow{\nabla} \mathcal{T}_0^1 \rightarrow$

Tensor Calculation Form

The set of **(tensor-calculation-)forms** is denoted by \mathcal{Form} and is defined to be the smallest set of tensor algebra morphisms such that

- every basic covariant tensor calculation is in \mathcal{Form} ;
- if A and B are in \mathcal{Form} and B may be composed with A , then $B \circ A$ is in \mathcal{Form} ;
- if $T \xrightarrow{f_i} T(i)$ for $1 \leq i \leq m$ are tensor algebra morphisms in \mathcal{Form} , then $T \xrightarrow{(f_1, \dots, f_m)} T(1) \times \dots \times T(m)$ is in \mathcal{Form} .

Manifest Covariance Theorem

Every tensor-calculation-form in \mathcal{Form} is covariant.

Proof Thought

Basic Proposition and structural induction.

“Electromagnetic four-Potential”

According to Wikipedia, the “differential of the electromagnetic potential” is

$$F_{ab} = \partial_a A_b - \partial_b A_a .$$

This expression for F_{ab} is *not* manifestly covariant. But by covariant differentiation (Willmore’s Formula),

$$\partial_a A_b = \nabla_a A_b + \Gamma_{ab}^c A_c$$

$$\partial_b A_a = \nabla_b A_a + \Gamma_{ba}^c A_c$$

Therefore,

$$\begin{aligned} F_{ab} &= (\nabla_a A_b + \Gamma_{ab}^c A_c) - (\nabla_b A_a + \Gamma_{ba}^c A_c) \\ &= \nabla_a A_b - \nabla_b A_a \end{aligned}$$

Since Γ is assumed torsion-free. Therefore, F_{ab} is manifestly covariant.

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“Einstein Tensor”

$$\text{Ricci} := R$$

$$\text{Scalar} := \text{Ricci} = R$$

$$\text{Einstein} := \text{Ricci} - \frac{1}{2} \text{Scalar}$$

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Thank you for your attention to

- algebraic-string diagrams
- manifest-covariance theorem
- functorial and natural definitions of geometric object and the forms of calculation