

# **ADDENDA**

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## INTUITION

“260125-CURVATURE-TORSION Based on the style and notation of my slides [see upload] on parallelism, write additional slides to construct curvature and torsion tensors, with emphasis on geometric intuition but perfectly coordinated with the algebraic constructions.”

## response from gipity: Lie Bracket as the Closure-Defect of a Parallelogram

For smooth vector fields  $U \xrightarrow{X} \mathcal{T}_0^1 U$  and  $U \xrightarrow{Y} \mathcal{T}_0^1 U$ , the **Lie bracket**  $[X, Y]$  is the vector field that measures the failure of the infinitesimal  $X$ - $Y$  parallelogram to close.

In coordinates this is the usual commutator of derivations, but geometrically it is:

$$[X, Y](p) \approx \frac{(\text{flow}_Y \circ \text{flow}_X - \text{flow}_X \circ \text{flow}_Y)(p)}{\text{area}}$$

We use  $[X, Y]$  below as the *correction term* that removes non-tensorial dependence on extensions.

## response from gipity: Torsion Tensor →

For a connection  $\nabla : \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \rightarrow \mathcal{T}_0^1 U$ , the **torsion** is the map

$$T : \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \rightarrow \mathcal{T}_0^1 U, \quad T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

**Type statement:**  $T$  is a  $(1, 2)$  tensor field (it is  $C^\infty$ -linear in each input).

**Geometric intuition:** torsion measures the *twist defect* of the connection: even if you try to build a small parallelogram using  $X$  and  $Y$ , the transported edges do not meet in the “expected” way.

## response from gipity: Torsion as a Tensorial Correction

The raw commutator  $\nabla_X Y - \nabla_Y X$  is *not* tensorial in  $X, Y$  because  $\nabla$  differentiates its inputs.

Subtracting the Lie bracket removes the non-tensorial part:

$$T(X, Y) = (\nabla_X Y - \nabla_Y X) - [X, Y].$$

Thus  $T(X, Y)(p)$  depends only on  $X(p), Y(p)$  (not on how  $X, Y$  are extended away from  $p$ ).

**Special case:**  $\nabla$  is **torsion-free** iff  $T = 0$ .

## response from gipity: Curvature Tensor

For a connection  $\nabla$ , the **curvature** is the map

$$\mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \xrightarrow{R} \mathcal{T}_0^1 U$$

defined by the corrected commutator:

$$R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z.$$

**Type statement:**  $R$  is a  $(1, 3)$  tensor field (it is  $C^\infty$ -linear in  $X, Y, Z$ ).

**Blackboard proof!**

**Geometric intuition:** curvature measures the *rotation defect* of parallel transport around an infinitesimal loop spanned by  $X$  and  $Y$ .

## response from gipity: Curvature as Path-Dependence of Parallel Transport

Fix  $p \in U$  and  $Z_0 \in \mathcal{T}_0^1 U|_p$ .

Transport  $Z_0$  along two infinitesimal routes:

$$p \xrightarrow{X} p + \varepsilon X \xrightarrow{Y} p + \varepsilon X + \varepsilon Y$$

versus

$$p \xrightarrow{Y} p + \varepsilon Y \xrightarrow{X} p + \varepsilon Y + \varepsilon X$$

The discrepancy, normalized by the infinitesimal area, is:

$$R(X, Y)Z_0(p).$$

**Slogan:**

Curvature is the obstruction to “path-independence of parallel transport.”

## response from gipity: Curvature as a Composed Operator (Algebraic Construction)

Think of  $\nabla$  as a binary operator

$$\nabla : \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \rightarrow \mathcal{T}_0^1 U.$$

Then the three terms in  $R(X, Y)Z$  are three *typed compositions* built from connection/covariant-derivative and Lie bracket:

$$\nabla \circ (X, \nabla \circ (Y, Z)), \quad \nabla \circ (Y, \nabla \circ (X, Z)), \quad \nabla \circ ([X, Y], Z).$$

The curvature tensor is their alternating sum.

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The curvature is built from three morphisms of the same codomain  $\mathcal{T}_0^1 U$ :

$$\mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \times \mathcal{T}_0^1 U \xrightarrow{\nabla \circ (\pi_{100}, \nabla \circ (\pi_{010}, \pi_{001}))} \mathcal{T}_0^1 U$$

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From the construction:

- $R(X, Y) = -R(Y, X)$  (antisymmetry in the first two slots).
- $T(X, Y) = -T(Y, X)$  (antisymmetry).
- If  $\nabla$  is torsion-free, then  $\nabla_X Y - \nabla_Y X = [X, Y]$ .

In particular, “torsion-free” means:

“Commutator of covariant derivatives reproduces the Lie bracket.”

## response from gipity: Connecting Back to Parallel Transport

Your theorem gives the parallel transport isomorphisms

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**Curvature says:** for a small loop  $\gamma$  based at  $\rho$ ,

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**Torsion says:** the “parallelogram” built from  $X$  and  $Y$  fails to close in the way a torsion-free connection predicts.

## Lemma

The assignment  $U \mapsto \mathcal{T}U, f \mapsto Jf$  defines a functor  $\text{Dom} \xrightarrow{\mathcal{T}} \text{Alg}$ ; that is, for  $U \xrightarrow{f} V \xrightarrow{g} W$  we have

$$\mathcal{T}(g \circ f) = \mathcal{T}g \circ \mathcal{T}f, \quad \mathcal{T}(\text{id}_U) = \text{id}_{\mathcal{T}U}.$$

At the conceptual level, the pushforward  $f_* == \mathcal{T}f$  is the linear map whose coordinate representation is the Jacobian  $Jf$ .

Lemma 

If  $g$  is a Riemannian metric. then for any diffeomorphism  $f : U \rightarrow \bar{U}$ , the index-lowering and index-raising maps

$$b : \mathcal{T}_0^1 U \rightarrow \mathcal{T}_1^0 \bar{U}, \quad \# : \mathcal{T}_1^0 U \rightarrow \mathcal{T}_0^1 \bar{U}$$

are covariant:

$$f_*(bX) = b(f_*X), \quad f_*(\#\alpha) = \#(f_*\alpha).$$

## Theorem

Let  $U \subset \mathbb{R}^n$  have a connection  $\nabla$ , and let  $\gamma : (a, b) \rightarrow U$  be a curve. Fix  $t_0 \in (a, b)$  and a vector  $v_0 \in \mathcal{T}_0^1 U|_{\gamma(t_0)}$ . Then there exists a unique parallel vector field  $X$  along  $\gamma$  such that  $X(t_0) = v_0$ .

## Theorem ←

If  $f : U \rightarrow \bar{U}$  is a diffeomorphism between domains with connections  $\nabla$  on  $U$  and  $\bar{\nabla}$  on  $\bar{U}$ , and if  $\gamma : (a, b) \rightarrow U$  is a curve and  $f \circ \gamma : (a, b) \rightarrow \bar{U}$  its image, then for all  $t_0, t_1 \in (a, b)$ ,

$$f_* \circ P_{\gamma, t_0}^{t_1} = P_{f \circ \gamma, t_0}^{t_1} \circ f_*$$

as maps  $\mathcal{T}U|_{\gamma(t_0)} \rightarrow \mathcal{T}\bar{U}|_{f(\gamma(t_1))}$ .

## Theorem

For each fixed  $Z$ , the map  $(X, Y) \mapsto R(X, Y)Z$  is  $C^\infty(U)$ -bilinear. Similarly, for each fixed  $X, Y$ , the map  $Z \mapsto R(X, Y)Z$  is  $C^\infty(U)$ -linear. Consequently,  $R$  defines a tensor field  $R \in \mathcal{T}_3^1 U$ .

## Microlect Specification

$$\text{LAW-OF-PHYSICS} = (K, C)$$

in which  $K$  is a finite set of key-terms and  $C$  is a minimal but adequate set of core sentences.

## Key Terms $K$

Structural	Covariance	Empirical	Ontological
law of physics	covariance	measurement	field
equation	invariance	observable	state
operator	diffeomorphism	evaluation	system
constraint	naturality	unit	model
kernel	transformation	dimension	
composition		experimental context	
morphism			

**Structural** terms determine the syntactic form of laws; **Covariance** terms govern admissible transformations; **Empirical** terms anchor laws to measurement; **Ontological** terms specify what laws are *about*.

## C1 — Law as Constraint

A **law of physics** is an **equation** that expresses a **constraint** on physically admissible **states** or **fields**.

## C2 — Operator Form

Every law of physics can be written as the vanishing or equality of an **operator** formed by composing admissible mathematical operations acting on physical quantities.

## C3 — Kernel Formulation

A law of physics asserts that realized physical states lie in the **kernel** of a specified operator or morphism.

## C4 — Dimensional Typing

All quantities appearing in a law of physics must be **dimensioned**, and the equation must be syntactically well-typed with respect to **units**.

## C5 — Measurement Interface

A law of physics must admit an **evaluation** via **measurement**, through observable quantities defined within an experimental context.

## C6 — Locality

Laws of physics apply **locally**, constraining the behavior of physical quantities in arbitrarily small regions of spacetime.

## C7 — Covariance

A law of physics must be **covariant**, retaining its form under admissible changes of coordinates, frames, or representations.

## C8 — Counterfactual Stability

A law of physics supports **counterfactual** reasoning: variations in conditions lead to predictable variations in outcomes.

## C9 — Compositional Closure

Laws of physics are closed under **composition**, allowing complex laws to be constructed from simpler lawful operations.

## C10 — Model Independence

A law of physics constrains a class of **models** rather than exhaustively describing any single physical system.

- No core sentence is derivable from the others alone.
- Removing any core sentence breaks the empirical or syntactic role of laws.
- The closure of  $\mathcal{C}$  generates standard distinctions:
  - ▶ law versus solution
  - ▶ law versus measurement
  - ▶ covariance versus coordinate dependence

## Law of Physics

A **law of physics** is a covariant, dimensionally-typed operator constraint that specifies physically admissible states and whose terms admit empirical evaluation under counterfactual variation.

- The functors  $L$  and  $R$  instantiate admissible tensor products from  $K$ .
- Natural transformations realize C2, C3, and C7.
- Measurement maps instantiate C5 without entering the law itself.

## INTUITION ←

According to Abraham-Marsden, “Foundations of Mechanics,” “1.4.1 Definition. Let  $S$  be a set. A local chart on  $S$  is a bijection  $\varphi$  from a subset  $U$  of  $S$  to an open subset of some (finite-dimensional, real) vector space  $F$ . An atlas on  $S$  is a family  $\mathbb{Q}$  of charts  $\{(U_i, \varphi_i) : i \in I\}$  such that (MA1)  $S = \cup \{U_i \mid i \in I\}$ ; (MA2) Any two charts in  $\mathbb{Q}$  are compatible in the sense that the overlap maps between members of  $\mathbb{Q}$  are  $C^\infty$  diffeomorphisms: for two charts  $(U_i, \varphi_i)$  and  $(U_j, \varphi_j)$  with  $U_i \cap U_j \neq \emptyset$  we form the overlap maps:  $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} \mid \varphi_i(U_i \cap U_j)$ , where  $\varphi_i^{-1} \mid \varphi_i(U_i \cap U_j)$  means the restriction of  $\varphi_i^{-1}$  to the set  $\varphi_i(U_i \cap U_j)$ . We require that  $\varphi_i(U_i \cap U_j)$  is open in  $F_i$ , and that  $\varphi_{ji}$  be a  $C^\infty$  diffeomorphism. Two atlases  $\mathbb{P}_1$  and  $\mathbb{Q}_2$  are equivalent iff  $\mathbb{Q}_1 \cup \mathbb{Q}_2$  is an atlas.”

## INTUITION, continued

“A differentiable structure  $\mathcal{G}$  on  $S$  is an equivalence class of atlases on  $S$ . The union of the atlases in  $\mathcal{G}$ ,  $\mathcal{Q}_{\mathcal{G}} = \cup\{Q \mid Q \in \mathcal{G}\}$  is the maximal atlas of  $\mathcal{F}$ , and a chart  $(U, \varphi) \in \mathcal{Q}_{\mathcal{G}}$  is an admissible local chart. If  $\mathcal{Q}$  is an atlas on  $S$ , then the union of all atlases equivalent to  $\mathcal{Q}$  is called the differentiable structure generated by  $\mathcal{Q}$ .”

### Intuition:

Can the idea of “equivalence class of atlases” be replaced by starting with an arbitrary atlas and closing it under the operation of adding compatible local charts? (2) Isn't  $S$  a colimit in a category of local charts in a maximal (i.e., closed) atlas?

## Theorem:

- **(1) Colimit Theorem** From a *closed* (maximal) smooth atlas, build a diagram of domains and overlap maps whose colimit is the manifold.
- **(2) Dom-Theorem** Reformulate smooth manifolds as certain sheaves on the site  $(\mathbb{D}om, \text{open covers})$ , i.e. via a functor.

## Caution (where colimits live)

To make “gluing” literally a colimit with the desired universal property, it is convenient to work in a category where such colimits exist (e.g.  $\mathbf{Sh}(\mathbb{D}om)$ , diffeological spaces, or a suitable category of smooth spaces). Under mild extra hypotheses one recovers an honest smooth manifold.

## Closed (“Maximal”) Atlas

An atlas  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$  on a set  $\mathcal{S}$  is *closed* (maximal) if it contains every chart  $(U, \varphi)$  that is  $C^\infty$ -compatible with all charts in  $\mathcal{A}$ .

Equivalently,  $\mathcal{A} = \overline{\mathcal{A}_0}$  is the closure of some initial atlas  $\mathcal{A}_0$  under adding compatible charts.

If  $(U_i, \varphi_i)$  with  $\varphi_i : U_i \rightarrow V_i \subset \mathbb{R}^n$  is a bijection onto an open set  $V_i$ , define for  $(i, j)$ :

$$V_{ij} := \varphi_i(U_i \cap U_j) \subset V_i$$

$$\tau_{ji} := \varphi^{-1} \circ \varphi : V_{ij} \cong V_{ji}.$$

## Recall: Category of Domains

$\mathbb{D}om$  is the category whose objects are open subsets  $U \subset \mathbb{R}^n$  for varying  $n \geq 0$ , and whose morphisms are smooth maps.

## Open-Cover Grothendieck Topology

A family  $\{U_\alpha \rightarrow U\}$  in  $\mathbb{D}om$  is a **covering** if it is induced by an open cover  $U = \bigcup_\alpha U_\alpha$  (with each  $U_\alpha \subset U$  open and maps the inclusions).

## Representables

Each  $U \in \mathbb{D}om$  defines a representable presheaf  $h_U := \mathbb{D}om(\_, U)$ .  
These are the “basic charts” on the sheaf side.

## Open-Cover Grothendieck Topology

From the domain images  $\{V_i\}_{i \in I} \subset \mathbb{R}^n$  and transition diffeomorphisms  $\tau_{ji} : V_{ij} \rightarrow V_{ji}$ , form the groupoid diagram (in  $\mathbf{Dom}$ , or after embedding  $\mathbf{Dom} \hookrightarrow \mathbf{Sh}(\mathbf{Dom})$ ):

$$G_1 \rightrightarrows G_0, \quad G_0 := \coprod_{i \in I} V_i, \quad G_1 := \coprod_{(i,j) \in I^2} V_{ij}.$$

The two arrows  $s, t : G_1 \rightrightarrows G_0$  are:

$$s|_{V_{ij}} : V_{ij} \hookrightarrow V_i \subset G_0, \quad t|_{V_{ij}} : V_{ij} \xrightarrow{\tau_{ji}} V_{ji} \hookrightarrow V_j \subset G_0.$$

### Triple overlap condition

Compatibility of charts implies  $\tau_{ii} = 1$  and  $(\tau_{ki} = \tau_{kj} \circ \tau_{ji}$  on triple overlaps, so  $G_1 \rightrightarrows G_0$  is an *effective* gluing relation.

## Theorem 1

If  $\mathcal{A}$  is a closed smooth atlas with domain data  $\{V_i, V_{ij}, \tau_{ji}\}$  as above, and diagram  $G_1 \rightrightarrows G_0$ , then in the category of sheaves  $\mathbf{Sh}(\mathbb{D}om)$  (open-cover topology), the colimit

$$\mathcal{M} := \operatorname{coeq}(h_{G_1} \rightrightarrows h_{G_0}) \simeq \operatorname{colim}(h_{G_1} \rightrightarrows h_{G_0})$$

exists and has the universal property: for any  $X \in \mathbf{Sh}(\mathbb{D}om)$ ,

$$\mathbf{Sh}(\mathbb{D}om)(\mathcal{M}, X) \cong \left\{ \{f_i : h_{V_i} \rightarrow X\}_{i \in I} : f_j \circ h_{\tau_{ji}} = f_i \text{ on } h_{V_{ij}} \right\}.$$

Moreover, if the usual separation/size hypotheses hold (e.g. the glued space is Hausdorff and second countable), then  $\mathcal{M} \cong h_M$  for a smooth manifold  $M$ , and the maps  $V_i \rightarrow M$  are charts whose maximal closure recovers  $\mathcal{A}$ .

## Theorem 2

For smooth manifold  $M$ , define presheaf functor

$$h_M : \mathbb{D}om^{op} \rightarrow \mathbb{S}et, \quad h_M(U) := C^\infty(U, M).$$

$h_M$  is a sheaf on  $(\mathbb{D}om, \text{open-covers})$ . The assignment

$$\mathbf{Man} \hookrightarrow \mathbf{Sh}(\mathbb{D}om), \quad M \mapsto h_M$$

is fully faithful: smooth maps  $M \rightarrow N$  correspond bijectively to natural transformations  $h_M \rightarrow h_N$ .

## Theorem 3

A sheaf  $F \in \mathbf{Sh}(\mathbb{D}om)$  is isomorphic to  $h_M$  for some smooth manifold  $M$  if and only if there exists a family of maps from representables

$$\{h_{V_i} \rightarrow F\}_{i \in I}$$

such that:

- **(Atlas / cover)** the induced map  $\coprod_i h_{V_i} \rightarrow F$  is an epimorphism of sheaves (locally surjective).
- **(Representable overlaps)** each fiber product  $h_{V_i} \times_F h_{V_j}$  is representable by some domain  $V_{ij} \in \mathbb{D}om$ .
- **(Local diffeomorphism data)** the two projections  $V_{ij} \rightrightarrows V_i, V_j$  correspond to open embeddings and transition diffeomorphisms satisfy the condition on triple overlaps.
- **(Separation/size)** the resulting glued space is Hausdorff and second countable.

In this case,  $F$  is the colimit of the associated Čech groupoid diagram from the atlas, as in Theorem 1.

## Equivalence with the “closed atlas on a set” definition

- From a closed atlas  $\mathcal{A}$  on a set  $S$ , build  $\mathcal{M} \in \mathbf{Sh}(\mathbb{D}om)$  by the coequalizer/colimit construction (Theorem 1).
- If the separation/size axioms hold,  $\mathcal{M} \cong h_M$  for a manifold  $M$ , and the induced maximal atlas on  $|M|$  agrees with  $\mathcal{A}$ .
- Conversely, given a manifold  $M$ , any atlas  $\{(U_i, \varphi_i)\}$  gives a representable cover  $\{h_{V_i} \rightarrow h_M\}$ , and its Čech diagram colimit recovers  $h_M$ .

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