Non-Compact Proofs

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The starting point

Hilbert's consistency program was abandoned in the 1930s when Gödel Second Incompleteness Theorem, G2, was interpreted as yielding the Unprovability of Consistency Thesis, UCT:

> *"There exists no consistency proof of a system that can be formalized in the system itself"* (Encyclopædia Britannica).

UCT implies that PA cannot prove of its own consistency, hence a proof of consistency of any stronger theory (like Set Theory).

Main findings in brief

It is well-known that PA is not finitely axiomatizable, but a proof of a single formula is **compact**, i.e., involves only a finite number of axioms. However, **some mathematical proofs are not compact**, e.g., the proof of Induction in PA, Mostowski Reflexivity Theorem, etc., require unbounded access to the axioms.

We address a fundamental blind spot in proof theory[1]: while G2 prohibits compact proofs of consistency within a system, it does not rule out non-compact ones. This allows for a formal proof of PA-consistency within PA by formalizing "selector proofs" that have been used tacitly in mathematics for decades.

This proof refutes UCT and removes the principal roadblock of the consistency studies initiated by Hilbert.

[1]The blind spot is what one does not see and what one is not even conscious of not seeing. (J.-Y. Girard)

Peano Arithmetic PA

Peano Arithmetic PA is a formal first-order theory containing constant 0, the successor function $'$, and all primitive recursive functions with their defining identities.

In addition, PA has the standard Induction schema: for each formula $\varphi(x)$, it is postulated that:

$$[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))] \rightarrow \forall x \varphi(x).$$

PA represents all conventional computations, does not use higher order or set-theoretic principles.

Models of arithmetic

The standard model of PA is the set

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots$$

with standard operations of addition and multiplication. It is easy to show that there are (countable) models of PA not isomorphic to the standard one. Take a fresh constant c and consider a theory

$$\widetilde{\mathsf{PA}} = \mathsf{PA} \ \cup \{c > 0,\ c > 1,\ c > 2,\ \ldots\}.$$

$\widetilde{\mathsf{PA}}$ is consistent since each of its finite subsystems is (obviously) consistent, hence $\widetilde{\mathsf{PA}}$ has a model $\mathcal{M}$, which is a model for PA with "infinite" numbers, non-isomorphic to the standard model:

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \qquad \cdots \longrightarrow \underset{c}{\bullet} \longrightarrow \cdots \ .$$

Numerals vs. natural numbers

For the purposes of Hilbert's consistency program, natural numbers are represented constructively as PA-numerals[2]

$$0, 0', 0'', 0''', \ \ldots \ .$$

Each standard number is represented by a numeral, but standard numbers as a set are not definable in PA. Gödel's numbering codes syntactic objects – formulas, finite sequences, *etc.* – by standard numbers.

The principal difference between the informal arithmetic and the formal arithmetic PA is in quantification. Since standard natural numbers cannot be defined in PA, *"for all natural numbers n ..."* is replaced in PA by the formal quantifier "$\forall x \ \ldots$" which, by definition, refers to all elements of a given, possibly non-standard, model: this makes $\forall x \varphi$ stronger than *"for all natural numbers n,* $\varphi(n)$*."*

G2 lives in this gap!

[2] cf. Hilbert strokes, Zermelo, von Neumann, Church numerals, *etc.*

What we are actually doing: mathematician's view

We assume math and logic within the standard university curriculum we teach: derivations, models, soundness, completeness, *etc.* A question of

$$\textit{whether consistency of } \mathsf{PA} \textit{ can be proved by means of } \mathsf{PA} \qquad (1)$$

is a conventional math problem. If nothing else, (1) is a typical problem of what can be done with limited tools, akin to doubling the cube using only a compass and straightedge.

We provide the affirmative answer to (1).

What is a formalization

i. **Direct formalization**. Let H be a contentual arithmetical property. Consider a formalization procedure of converting H into a formal PA-object: the primitive recursive operations, logical connectives are formalized as is, and *"for any natural number n"* is formalized as "$\forall x$"[3]. If this procedure succeeds on H, we call the resulting $f(H)$ the *direct formalization of* H; $f(H)$ can be a PA-formula, or a set of PA-formulas.

ii. **Gödelian formalization** is a standard arithmetization (syntactic objects are assigned numerical codes, operations on objects become functions on codes) followed by the direct formalization (i) in PA.

iii. **Formalization of reasoning** about math objects is a standard step-by-step conversion of informal reasoning into formal derivations in PA about Gödelian codes.

Formalizations (i) – (iii) are well-understood when used within commonly accepted boundaries. All formalizations in this work fall into such a noncontroversial category.

[3] Mind the domain violation.

Why do we formalize proofs?

Suppose we are interested in

$$\textit{whether a contentual property } H \textit{ of natural numbers holds.} \qquad (2)$$

Suppose also that its formalized version $f(H)$ is a PA formula. Proving $f(H)$ is PA can answer the original question about the validity of H.

A. If PA proves $f(H)$ then $f(H)$ holds in all models of PA, including the standard model, which gives an affirmative answer to (2).

B. If PA proves the negation of $f(H)$, then $f(H)$ fails in each model of PA including the standard model, which gives a negative answer to (2).

C. However, if neither **A** nor **B** holds, then we have no answer to (2) and must rely on other tools.

In addition, if a contentual proof is formalized in PA, then this proof does not use the principles outside PA.

So, we formalize proofs for **verification** and **assumptions checking**.

Formalizing Consistency in PA: arithmetization

The mathematical formulation of PA-consistency,

$$\textit{no } D \textit{ is a } \mathrm{PA}\textit{-derivation of } (0=1), \quad (3)$$

uses a universal quantifier over finite sequences D of formulas, not in the language of arithmetic, hence some Gödel coding is required.

In particular, Gödel constructed a primitive recursive arithmetical formula $\mathit{Consistent}(x)$ such that for any numeral n, $\mathit{Consistent}(n)$ states that

$$\text{“}\textit{n is not a code of a } \mathrm{PA}\textit{-derivation of } (0=1)\text{.”}$$

Arithmetized PA-consistency is the contentual property

$$\textit{for any natural number } n,\ \mathit{Consistent}(n), \quad (4)$$

still not in the language of PA since quantifiers over standard numbers are not expressible in PA.

Notations: $x{:}y$ is the proof predicate “x *is a* PA-*derivation of* y,” $\perp$ is $(0=1)$, $\mathit{Consistent}(x)$ is $\neg x{:}\perp$. $\Box(y)$ is the provability predicate $\exists x(x{:}y)$. We do not distinguish between X, its code $\ulcorner X \urcorner$, and its numeral $\overline{\ulcorner X \urcorner}$.

The consistency formula $\mathsf{Con}_{\mathsf{PA}}$

This is an important bifurcation point, which has been overlooked.

Traditional route: ignore the domain violation and assume that the mathematical statement of consistency (4) is represented in PA by a formula $\mathsf{Con}_{\mathsf{PA}}$:

$$\mathsf{Con}_{\mathsf{PA}} = \forall x\ \mathit{Consistent}(x) = \forall x(\neg x{:}\bot). \tag{5}$$

Gödel's Second Incompleteness Theorem.
If PA *is consistent, then* PA *does not prove* $\mathsf{Con}_{\mathsf{PA}}$*.*

Corollary. *There is a model* $\mathcal{M}$ *of* PA *in which* $\mathsf{Con}_{\mathsf{PA}}$ *is false.*

Gödel's monster. There is a "proof" of $(0{=}1)$ in $\mathcal{M}$. However, such a "proof" cannot be a numeral $n = 0, 1, 2, \ldots$, i.e., it is non-standard in $\mathcal{M}$.

Consequences of the domain violation in $\mathsf{Con}_{\mathsf{PA}}$

As we saw, there are models of PA with inconsistent proofs. However, **all such "bad" proofs turned out to be nonstandard**, hence G2 does not appear to be about real PA-derivations which are all finite and which the original contentual consistency question has been all about.

> *Internalization appears to distort the intrinsic nature of consistency and makes it unprovable since the language of* PA *is too weak to sort out fake proof codes.*

A not too remote analogy

Imagine we wanted to prove that

no dollar bill displays Mickey Mouse. (6)

The Certification Department by "dollar bills" understands both real US dollar bills and toy "monopoly money" that may feature Mickey Mouse images. So, it cannot issue a blanket certificate that all dollar bills are Mickey Mouse-free, but it can clear each given US dollar bill.

We equivalently reformulated the request: to certify that

> *the US dollar bills of all eleven legal denominations $1, $2, ..., $10,000 of all versions are Mickey Mouse-free.*

We got an instant certification.

By definition, the consistency is a serial property

The consistency property (4):

for any natural number n, Consistent(n),

is **provably in** PA equivalent to the serial property[4]:

$$Consistent(n),\ \ n = 0, 1, 2, \ldots\ . \tag{7}$$

which is strictly weaker in PA than the consistency formula $\mathsf{Con}_{\mathsf{PA}}$.

Serial properties are common objects in mathematics and logic:

- tautologies in PA,
- PA - axioms,
- ZF - Comprehension/Separation Schema,
- Reflection Schemas,
- *etc., etc.*

[4] A serial property is a primitive recursive series of formulas $F_1, F_2, F_3, \ldots$

Compact and Non-Compact Proofs

The usual way to prove “for all n, $F(n)$” in mathematics is

given an arbitrary n provide an argument $\mathcal{A}(n)$ concluding $F(n)$.

Imagine that $\mathcal{A}(n)$ **requires more and more new axioms with the growth of** n? Such “non-compactness” appears, e.g., in the proof of the Induction Principle in PA, which requires an unrestricted access to the induction schema[5]. An easy proof-theoretic analysis shows that the **conventional derivations in** PA **cover only compact proofs**, i.e., proofs that fit into some finite fragment of PA.

This answers the question: which proofs are actually banned by G2?

G2 shows that Consistency does not have a compact proof, but we are free to search for non-compact proofs.

[5] Other examples: MRT, a hypothetical non-compact proof of the twin primes conjecture.

Summary: arguments of why the UCT is not justified

Conceptual. Domain violation in the consistency formula $\mathrm{Con}_{\mathsf{PA}}$: "$\forall x$" ranges over domains of PA-models, mostly nonstandard, and as such is not a good replacement of *"for all natural numbers n."*

Logical. $\mathrm{Con}_{\mathsf{PA}}$ is strictly stronger then "PA is consistent" in PA, hence UCT is a product of *the strengthening fallacy*: UCT uses the unprovability of $\mathrm{Con}_{\mathsf{PA}}$ to claim the unprovability of "PA is consistent."

Mathematical. G2 rules out only compact proofs of consistency, but does not prohibit non-compact proofs, hence does not justify UCT.

Now we are going to show that **UCT is false** by providing a (non-compact) proof of PA-consistency and formalizing it in PA.

The question persists: is PA-consistency provable in PA?

Since PA-consistency is not a single formula but a series of arithmetical formulas (a serial property), the first step should be developing a rigorous notion of a proof of a serial property in PA.

Actually, proofs of serial properties have long been part of contentual mathematical practices, waiting for its rigorous definition.

"Instance provability" alone is too weak

A naive "instance provability" approach to proving a serial property $\mathcal{F} = \{F_0, F_1, \ldots, F_n, \ldots\}$ by means of a theory T

$$\textit{for each } n,\ T \textit{ proves } F_n \tag{8}$$

is too weak since it does not address the issue of the proof of (8) in T.

For example, the consistency proof for PA via truth in the standard model yields instance provability of the consistency property:

> *All theorems of* PA *are true in the standard model and* $\perp$ *is not true. Therefore, for each* n*, Consistent*(n) *holds and hence is provable in* PA *as a true primitive recursive statement.*

However, it is not a proof in PA since the notion "true in the standard model" is not formalizable in PA. So, in addition to "instance provability" of $\mathcal{F}$ in T, some sort of verification of (8) by means of T is also needed.

A proof of a serial property: Hilbert's take

Hilbert's understanding of consistency proofs (Richard Zach):

> *"What is required for a consistency proof is an operation which, given a formal derivation, transforms such a derivation into one of a special form, plus proofs that the operation in fact succeeds in every case and that proofs of the special kind cannot be proofs of an inconsistency."*

In a slightly generalized form, a Hilbertian consistency proof is

(i) an operation that, given a derivation D, yields a proof that D is free of contradictions,

(ii) a proof that (i) works for all inputs D.

Hilbert's approach in a general setting: *selector proofs*

The following definition represents Hilbert's ideas in a general setting.

A proof of a serial property $\mathcal{F} = \{F_0, F_1, \ldots, F_n, \ldots\}$ *in a theory* T is a pair of

(i) *selector*: an operation[6] that given n provides a proof of F_n in T;

(ii) *verifier*: a proof in T that the selector does (i).

We call such pairs (i) and (ii) *selector proofs*.

[6] For the purposes of this work, selectors are explicit primitive recursive operations but this can be naturally extended to other provably total functions.

The fundamental properties of selector proofs in PA

- Selector proofs absorb the usual proofs as special cases.
- Selector proofs are as good as the usual proofs in establishing arithmetical truths: if a serial property $\mathcal{F}$ is selector provable, then each of the F_n's is provable in the usual sense.
- Selector proofs do not extend the power of PA to prove formulas, they just reveal the ability of PA to also prove serial properties.
- Selector proofs have been tacitly and widely adopted in contentual mathematics.

Example 1

Complete Induction principle, $\mathcal{CI}$: *for any formula* ψ,

$$\textit{if for all } x\ [\forall y < x\ \psi(y) \textit{ implies } \psi(x)], \textit{ then } \ \forall x \psi(x).$$

Complete Induction for PA is provable by means of PA. Here is a textbook proof of $\mathcal{CI}$: *apply the usual* PA-*induction to* $\forall y < x\ \psi(y)$ *to get the* $\mathcal{CI}$ *statement* $\mathcal{CI}(\psi)$ *for* ψ.

This is a selector proof which, given ψ selects a derivation of $\mathcal{CI}(\psi)$ in a way that provably works for any input ψ.

Note a clear non-compact character of this proof.

Example 2

The product of polynomials is a polynomial.[7]

Here is its standard mathematical proof: *given a pair of polynomials f, g, using the well-known formula, calculate coefficients of the product polynomial $p_{f\cdot g}$, and prove in arithmetic that*

$$f \cdot g = p_{f\cdot g}. \tag{9}$$

This is a selector proof: for each f, g, it finds a proof of (9) in PA.

[7] A polynomial is a term $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where each a_i is a numeral and x a variable. In $f \cdot g$, "$\cdot$" stands for the usual PA multiplication.

Example 3

Take the double negation law, DNL[8] in arithmetic: *for any formula* X*,*

$$X \leftrightarrow \neg\neg X. \tag{10}$$

The standard proof of DNL in PA is *for given* X*, build the usual logical derivation* $D(X)$ *of* (10) *in* PA*.*

This is a selector proof which builds an individual PA-derivation for each instance of DNL in a way that provably works for any input X. This proof can be easily formalized in PA.

[8] or any other tautology containing propositional variables.

Ubiquitous selector proofs

Selector proofs have been used as proofs in arithmetic of serial properties

$$\{F(u)\}$$

in which u is a syntactic parameter (ranging over terms, formulas, derivations, etc.) and $F(u)$ is an arithmetical formula for each u.

Loosely, selector proofs naturally appear, e.g., when the formalization of a property of interest S is not a single arithmetical formula, or when PA-proofs of instances $S(n)$ are non-compact.

Within this tradition, we have to accept and study selector proofs as legitimate logic objects. In this case, the proof of PA's consistency within PA (below) is a natural consequence.

Selector proof vs. single formula consistency proofs

Let Con_T be the standard consistency formula for a theory $T \supseteq \mathrm{PA}$. Consider theories:

$$\mathrm{PA}_0 = \mathrm{PA}, \qquad \mathrm{PA}_{i+1} = \mathrm{PA}_i + \mathrm{Con}_{\mathrm{PA}_i}, \qquad \mathrm{PA}^\omega = \bigcup \mathrm{PA}_i.$$

Consider a folklore consistency proof[9] of PA^ω by means of PA^ω:

> *Let D be a derivation in PA^ω. Find n such that D is a derivation in PA_n. $\mathrm{Con}_{\mathrm{PA}_n}$ – one of the postulates of PA^ω – implies that D does not contain $\bot$.*

This is a (non-compact) selector proof of consistency of PA^ω in PA^ω. The selector that, given D, computes the code of a PA^ω-derivation of "D does not contain $\bot$" is straightforward, as well as its verification.

By G2, PA^ω does not prove $\mathrm{Con}_{\mathrm{PA}^\omega}$ but this fact does not harm the consistency proof which actually does not derive $\mathrm{Con}_{\mathrm{PA}^\omega}$.

[9] Many thanks to Moshe Vardi for reminding about this example.

What does it mean to prove the consistency of T in T?

People new to logic often ask a naive question: what is the point of proving the consistency of a theory T in T itself, since if T is inconsistent, the answer would be vacuously affirmative.

The Hilbertian reply: "a proof of T-consistency in T" is in fact a pair:

i) a contentual mathematical proof of T-consistency;

ii) a Gödelian formalization of (i) as a formal derivation in T for the assumptions checking.

This is exactly what we are doing.

(I) We provide a contentual non-compact proof of PA-consistency;

(II) We build a Gödelian step-by-step formalization of (I) in PA for the assumptions checking.

Explicit Reflection

While Gödel suggested the concept of explicit reflection a public lecture in Vienna in 1938, this work remained unpublished for nearly 60 years until 1995. During that gap, the explicit reflection principle

$$t{:}F \rightarrow F \tag{11}$$

was independently rediscovered by Artemov and Strassen in 1992. These works identified that while the implicit reflection principle

$$\Box F \rightarrow F$$

is unprovable in PA for some F, the explicit version (11) is provable within PA. In the modern terminology, (11) has a non-compact selector proof in PA.

Why MRT does not prove PA-consistency in PA

Let $\mathsf{PA}_{\restriction n}$ be the fragment of PA with the first n axioms.

The Mostowski Reflexivity Theorem, MRT, states:

For each natural number n, $\mathsf{PA} \vdash \mathsf{Con}_{\mathsf{PA}_{\restriction n}}$.

In order to get the PA-consistency:

For each natural number n, $\mathsf{Con}_{\mathsf{PA}_{\restriction n}}$,

one needs to strip "$\mathsf{PA} \vdash$" in this MRT formulation, which cannot be done within PA due to the failure of the implicit reflection.

So MRT has not been considered a proof of PA-consistency, let alone a proof of PA-consistency in PA. However, the proof of MRT itself is a non-compact selector proof which, with proper adjustments, becomes an integral part of our PA-consistency proof within PA.

The contentual consistency proof

Step 1 - Follow the explicit steps of the MRT proof (and resist the temptation to simplify them to their implicit versions). Namely, from the proof of MRT, extract a primitive recursive function s(x), selector, that given n builds a derivation $s(n)$ of $\mathrm{Con}_{\mathrm{PA}\restriction n}$ in PA. Note that $\mathrm{Con}_{\mathrm{PA}\restriction n}$ implies "*D is not a proof of $\bot$*" for any specific derivation D in $\mathrm{PA}\restriction n$.

Step 2 - Use the explicit reflection to rid the interfering proof predicates. Given a PA-derivation D, find n such that D is a derivation in $\mathrm{PA}\restriction n$ (an easy primitive recursive procedure). By step 1,

$$s(n){:}\mathrm{Con}_{\mathrm{PA}\restriction n}$$

By the explicit reflection, we get

$$\mathrm{Con}_{\mathrm{PA}\restriction n},$$

hence

"*D is not a proof of $\bot$.*"

Step-by-step formalization of the consistency proof

No element in this consistency proof is outside the scope of PA and this whole proof is naturally internalized in PA. The reasoning leading to the claim *"D is not a proof of* $\bot$*"* is represented by a primitive recursive selector $S(x)$ which for a given Gödel number k of D returns a code of a PA-derivation of *"k is not a code of a proof of* $\bot$*"*.

The selector $S(x)$ for the whole consistency proof is a natural fusion of two selectors, $s(x)$ from the selector proof of MRT, and $e(x)$ from the selector proof of the explicit reflection in PA. So, we get the desired verification

$$\mathrm{PA} \vdash \forall x\ S(x){:}\neg x{:}\bot.$$

This completes the assumptions checking and certifies that **the given proof is a proof within** PA without any additional constructions or assumptions. Note that the given formalization of the consistency proof is not, and could not be, a PA-proof of the consistency formula $\mathrm{Con}_{\mathrm{PA}}$.

The failure of UCT in its naive Encyclopedia form

The Unprovability of Consistency Thesis in its naive encyclopedia form fails and should be corrected:

There exists a consistency proof of PA *that can be formalized in* PA.

What was a logical mistake in the standard justifiication of UCT? The Formalization Principle has been used in a too strong and incorrect form:

Any mathematical reasoning from axioms of a theory T proving a property H, can be formalized as a derivation of $f(H)$ in T.

This perspective suggests that the Hilbert Program wasn't "killed" by Gödel; it was simply forced to move beyond single-sentence goals. It implies that a system can be self-consistent and self-aware of that consistency, provided we don't force that awareness into a single "bottleneck" formula.

New reading of G2

These findings suggest new foundational reading of Gödel's Second Incompleteness Theorem:

The consistency of PA *is not provable within a finite fragment of* PA,

complemented with the positive message:

The consistency of PA *is provable within the whole* PA.

Not an epistemic failure, but a compression failure

This work reframes Gödel's Second Incompleteness Theorem not as an epistemic failure, but as a compression failure.

In this view, the "system" (PA) possesses the internal resources to verify the consistency of any specific derivation it can produce. However, it cannot "package" this infinite sequence of individual verifications into a single, finite sentence $\mathsf{Con}_{\mathsf{PA}}$ using the standard universal quantifier without accidentally including non-standard numbers – infinite "junk" values that Gödel's formula must account for, but which the actual system never produces.

Eliminating the “Reflection Tower”

Traditionally, proving consistency required stronger and stronger theories (a “reflection tower”). The aforementioned results show that a sufficiently rich theory like PA or ZF set theory can establish its own consistency without adding new axioms, effectively closing a principal roadblock in Hilbert's consistency program.

Closing the “Intuition” Gap in AI

These findings shift the “human vs. machine” debate by dismantling the most famous argument for human mathematical superiority: the Lucas-Penrose Thesis.

For decades, philosophers like J.R. Lucas and Roger Penrose argued that because humans can “see” the truth of a system’s consistency while the system itself (per Gödel) cannot, human reason must be non-algorithmic or “creative.”

Impact on foundations of verification

These findings free foundations of verification from some intrinsic impossibility limitations. Imagine that we want to verify the property

$$\forall x[t(x) = 0] \tag{12}$$

for some total computable term $t(x)$ by proving (12) in PA.

In the traditional G2 framework, in addition to a formal proof of (12) in PA, one needs some consistency assumptions about PA to conclude that $t(n)$ returns 0 for each $n = 0, 1, 2, \ldots$. Since, it was assumed that these additional assumptions could not be verified in PA, this left an annoying foundational loophole.

In our framework, PA proves its consistency, these additional meta-assumptions could be dropped, and proving (12) formally in PA is certified as a self-sufficient verification tool.

How far we can go with proving consistency in PA

Kurahashi and Sinclaire, 2019:

PA *cannot prove consistency of any theory* $T \supseteq \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ *by the given method without further modifications*.

Freund/Pakhomov and Gadsby, 2024/25:

However, PA *selector-proves consistency of* T *for some proper extensions* T *of* PA *in particular* $T = \mathsf{PA} +$ *slow consistency of* PA.

Beyong PA

Consider a theory $T \supseteq \mathrm{PA}$ in the language of PA. Does it prove self-consistency? Yes, it does, and the proof is basically the same.

Note that ZF is *essentially reflexive*. Namely, any theory $T \supseteq \mathrm{ZF}$ proves the standard consistency formula for each of its finite fragments $\{\varphi\}$; given φ we constructively build the proof of $\mathrm{Con}_{\{\varphi\}}$ in T.

This suggests a selector proof of consistency of any $T \supseteq \mathrm{ZF}$ in T.

As before, the mathematical value of such proofs depends on how much trust we invest into contentual T itself.

Any progress in Hilbert Program?

Before: *It was believed that Gödel's second incompleteness theorem prevented the proving of consistency of a system within that system. It was also thought that Hilbert's consistency program could not succeed because of that.*

After: *Gödel's second incompleteness theorem does not apply to Hilbert's approach to proving consistency how it was previously thought. Though the ϵ-substitution method for* PA *fails (by non-Gödelian reasons), some modifications of Hilbertian methods prove consistency of* PA *within* PA.

The limitation results by Kurahashi, Sinclaire, and Gadsby indicate that new ideas are needed for further progress in Hilbert's program.

Any changes in the traditional Proof Theory?

Existing methods and results remain valid, with minor cosmetic changes, e.g., "consistency of PA" should mean "consistency formula for PA."

A new topic, **proof theory of selector proofs**, is emerging, opening up new research opportunities.

The Foundations of Mathematics and its popularization activity require a major rethinking and revision, particularly the "Unprovability of Consistency" section. This should include adjusting Encyclopedia articles accordingly.

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