Graduate Center CUNY

Consistency of PA is a serial property, and it is provable in PA

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Proving consistency of PA in PA: what is the point?

PA is just an iconic case of "impossibility," with obvious generalizations.

We distinguish PA and its informal (contentual) version $\widehat{\text{PA}}$ which is a mathematical theory with the same axioms as PA. Here is a typical question when such a distinction is necessary.

> *Imagine a formal theory T proves its consistency. How can we claim that T is indeed consistent since if T were inconsistent, it would prove everything, including its consistency?*

Hilbert's approach was to find a contentual mathematical proof of the consistency of T by "trusted means" with a subsequent formalization of the proof in T. In the case of proving PA-consistency in PA, we are talking about a proof of PA-consistency in $\widehat{\text{PA}}$ which plays the role of a "trusted core," followed by a formalization of this proof in PA.

The widely accepted *Formalization Principle*, **FP** states that

> *any rigorous reasoning within* $\widehat{\text{PA}}$ *can be formalized in* PA.

PA-consistency and its Gödelization

Notations: $\overline{n}$ is a numeral $0^{'''\ldots(n\ times)}$, $\ulcorner X \urcorner$ is the Gödel number of X. We do not distinguish X, $\ulcorner X \urcorner$, and $\overline{\ulcorner X \urcorner}$ when safe. $\bot$ denotes $\overline{0}=\overline{1}$.

Standard definition of PA **consistency**:

$$\textit{for any derivation } D,\ D \textit{ is not a proof of } \overline{0}=\overline{1}. \tag{1}$$

Proof predicate: $x{:}y$ is a natural primitive recursive arithmetical formula

"x is a code of a proof of y."

Gödelization of (1):

$$\textit{for any numeral } \overline{n},\ \neg\overline{n}{:}\bot. \tag{2}$$

(1) and (2) are not in the language of PA.

Consistency scheme: the series $\mathrm{Con}^{S}_{\mathrm{PA}}$ of PA-formulas

$$\{\neg\overline{0}{:}\bot, \neg\overline{1}{:}\bot, \neg\overline{2}{:}\bot, \neg\overline{3}{:}\bot \ldots\}. \tag{3}$$

(3) consists of well-defined statements in the language of PA.

PA-consistency is provably equivalent to $\mathrm{Con}^S_{\mathrm{PA}}$

Proposition 1.

$$(1) \quad \Leftrightarrow \quad (2) \quad \Leftrightarrow \quad (3).$$

The proof of these equivalences is formalizable in PA *via standard Gödelization.*

Proof. The mathematical proof of these equivalences is straightforward. It can be formalized in PA in the standard fashion: finite objects (derivations, numerals) are represented by Gödel numbers, operations become natural p.r. terms, $\mathrm{Con}^S_{\mathrm{PA}}$ is coded by a natural p.r. term producing the Gödel number of $\neg \overline{n}{:}\bot$ given n, etc. We use the truth predicate Tr_1 for Σ_1 formulas to recover a p.r. formula φ from its code since $\mathrm{PA} \vdash \varphi \leftrightarrow Tr_1(\overline{\ulcorner \varphi \urcorner})$.

Corollary: PA*-consistency is adequately represented in the language of* PA *by the consistency scheme* $\mathrm{Con}^S_{\mathrm{PA}}$.

Consistency formula

The *provability predicate* $Pr(\varphi)$ is

$$\exists x(x{:}\varphi).$$

We will also short $Pr(\varphi)$ to $\Box\varphi$.

Consistency formula is the arithmetical formula $\mathrm{Con}_{\mathrm{PA}}$, is $\neg\Box\bot$, i.e.,

$$\neg Pr(\bot) \quad \text{or} \quad \forall x(\neg x{:}\bot).$$

A naive (and fundamental) question is whether $\mathrm{Con}_{\mathrm{PA}}$ is equivalent to the property of PA-consistency? Can such an equivalence be established by means of PA itself?

Consistency property vs. consistency formula

The consistency formula $\forall x \neg(x{:}\bot)$ in the PA-context is not a well-defined mathematical statement before the domain of $\forall x$ is specified.

The natural argument for why PA-consistency yields $\mathsf{Con}_{\mathsf{PA}}$ is

> *W.l.g., assume that Gödel numbering of derivations is surjective, i.e., each n is a code of some derivation. Suppose "not $\mathsf{Con}_{\mathsf{PA}}$." Then,* **for some standard** n, $\overline{n}{:}\bot$, *hence for some D, D is a derivation of contradiction.*

This argument assumes that "$\forall x$" ranges the standard natural numbers, which is not formalizable in PA hence requires meta-assumptions about PA (standard model) stronger than consistency.

So, the claim of the equivalence of PA-consistency and $\mathsf{Con}_{\mathsf{PA}}$ has no justification in PA. Moreover, we can show that in PA,

$\mathsf{Con}_{\mathsf{PA}}$ **is strictly stronger than** PA-**consistency.**

Con_{PA} is strictly stronger than PA-consistency

By Proposition 1, the scheme Con^S_{PA} is an equivalent representation of (1) and can be used to compare PA-consistency with Con_{PA}.

Proposition 2. Con_{PA} *is* **strictly stronger** *than* Con^S_{PA} *in* PA*.*

Proof. Indeed, it is easy to note that

$$\text{PA} + \text{Con}_{\text{PA}} \vdash \text{Con}^S_{\text{PA}} \quad \textit{but} \quad \text{PA} + \text{Con}^S_{\text{PA}} \not\vdash \text{Con}_{\text{PA}}.$$

So, Con_{PA} is **strictly stronger** in PA than the PA-consistency property. Consider the Unprovability of Consistency Thesis, UCT:

> *"There exists no consistency proof of a system that can be formalized in the system itself" (Encyclopædia Britannica).*

UCT is widely believed but has never been justified. Though, by the Second Gödel's Incompleteness Theorem G2, PA does not prove Con_{PA}, this does not yield the unprovability of PA-consistency.

This UCT story appears to be one epic blunder in foundations

The question persists: is PA-consistency provable in PA?

By Proposition 1, this reduces to

whether $\mathrm{Con}^S_{\mathrm{PA}}$ **is provable in** PA.

Since $\mathrm{Con}^S_{\mathrm{PA}}$ is not a single formula but a series of arithmetical formulas (a serial property), the first step should be developing a rigorous notion of a proof of a serial property in PA.

Fortunately, such a notion has long been part of contentual mathematical reasoning, waiting for a rigorous logic formalization.

A proof of a serial property: quantification is too strong

A common "overkill" approach to proving serial properties in PA:

> *Let* $\mathcal{F}$ *be* $\{F(0), F(1), \ldots, F(n), \ldots\}$ *for some arithmetical formula* $F(x)$*. Then "*$\mathcal{F}$ *is provable in* PA*" means*
>
> $$\mathrm{PA} \ \vdash \ \forall x F(x) \qquad\qquad (\dagger).$$

As we have already noticed, in PA, formula $\forall x F(x)$ can be strictly stronger than $\mathcal{F}$, since $\forall x F(x)$ immediately yields $\mathcal{F}$, but $\mathrm{PA} + \mathcal{F}$ does not necessarily yield $\forall x F(x)$.

So, using $\forall x F(x)$ in lieu of $\mathcal{F}$ is a strengthening fallacy and should not be applied to establishing unprovability of $\mathcal{F}$ in PA. Mathematically, this fallacy amounts to assuming the ω-rule which spills over PA.

and instance provability is too weak

A naive "instance provability" approach to proving a serial property $\mathcal{F} = \{F_0, F_1, \ldots, F_n, \ldots\}$ by means of S

$$\textit{for each } n, \; S \textit{ proves } F_n \tag{4}$$

is too weak since it does not address the issue of the proof of (4) in S. For example, the consistency proof for PA via truth in the standard model yields instance provability of the consistency property:

> *Let D be a formal derivation in* PA*. Since all formulas from D are true in the standard model and $\perp$ is not true, the latter is not in D. Therefore $\neg\overline{n}{:}\perp$ is true for each numeral $\overline{n}$, hence provable in* PA *as a true primitive recursive statement.*

However, it is not a proof in PA since the notion "true in the standard model" is not formalizable in PA. So, in addition to "instance provability" of $\mathcal{F}$ in S, some sort of verification of (4) by means of S is also needed.

A proof of a serial property: Hilbert's take

Here is Richard Zach's summary of Hilbert's understanding of consistency proofs ("Hilbert's program then and now" in Philosophy of Logic, 2007):

> *"What is required for a consistency proof is an operation which, given a formal derivation, transforms such a derivation into one of a special form, plus proofs that the operation in fact succeeds in every case and that proofs of the special kind cannot be proofs of an inconsistency."*

In a slightly generalized form, a Hilbertian consistency proof is

(i) an operation that, given D, yields a proof that D is free of contradictions,
(ii) a proof that (i) works for all inputs D.

Hilbert's approach in a general setting: *selector proofs*

The following definition represents Hilbert's ideas in a general setting.

A proof of a serial property $\mathcal{F} = \{F_0, F_1, \ldots, F_n, \ldots\}$ *in a theory* S is a pair of

(i) *selector*: an operation[1] that given n provides a proof of F_n in S;

(ii) *verifier*: a proof in S that the selector does (i).

We call such pairs (i) and (ii) *selector proofs*.

We will provide a body of examples that demonstrate that selector proofs have already been tacitly adopted by mathematicians. The time is ripe for logicians to catch up.

[1] For the purposes of this work, selectors are explicit primitive recursive operations but this can be naturally extended to other provably total functions.

Formalizing selector proofs in PA

Let

$$\mathcal{F} = \{F(u)\}$$

be a serial property with a syntactic parameter u such that for each u, $F(u)$ is an arithmetical formula. The natural formalization in PA of a given selector proof of $\mathcal{F}$ consisting of a selector and a verifier is:

i) An arithmetical term $s(x)$ formalizing the selector procedure which given $\ulcorner u \urcorner$ builds the code of some proof of $F(u)$.

ii) A PA-formalization of the verifier which is a PA-proof v of

$$\forall x\ s(x){:}F^{\bullet}(x)$$

for a natural "coding" term $F^{\bullet}(x)$ such that $F^{\bullet}(\ulcorner u \urcorner) = \ulcorner F(u) \urcorner$.

Hilbert meets Brouwer on consistency proofs

Consider the Brouwer-Heyting-Kolmogorov clause for

$$\forall x F(x). \tag{5}$$

In Kreisel's form, a *constructive proof of* (5) is a pair $\langle s, v\rangle$ where v is a proof that for each x, $s(x)$ is a proof of $F(x)$.

BHK proofs of universal statements are intrinsically selector proofs.

Why selector proofs are acceptable as proofs

Encyclopedia Britannica:

> *a proof is an argument that establishes the validity of a proposition.*

For example, a proof in PA satisfies this requirement: if PA proves F then F is a valid statement about integers.

Selector proofs appear to pass this test as well: as we show later, if PA (selector) proves a serial property $\mathcal{F}$, then each instance of $\mathcal{F}$ is provable in PA and hence $\mathcal{F}$ is a valid set of statements about integers.

This suggests that selector proofs constitute a sound conservative extension of the conventional notion of arithmetical proof of formulas on a new class of syntactic objects: serial properties.

Example 1

Complete Induction principle, $\mathcal{CI}$: *for any formula* ψ,

$$\textit{if for all } x\ [\forall y < x\ \psi(y) \textit{ implies } \psi(x)],\ \textit{then}\ \ \forall x \psi(x).$$

Complete Induction for PA is provable by means of PA. Here is a textbook proof of $\mathcal{CI}$: *apply the usual* PA-*induction to* $\forall y < x\ \psi(y)$ *to get the* $\mathcal{CI}$ *statement* $\mathcal{CI}(\psi)$ *for* ψ.

This is a selector proof which, given ψ selects a derivation of $\mathcal{CI}(\psi)$ in a way that provably works for any input ψ. Its formalization in PA:

- pick a natural primitive recursive selector term $s(x)$ which given ψ computes the code of a derivation in PA of $\mathcal{CI}(\psi)$;
- find an easy proof in PA (verifier) that $s(x)$ works for all inputs

$$\mathsf{PA} \vdash \forall x\ s(x) : CI^{\bullet}(x)$$

with $CI^{\bullet}(x)$ a natural p.r. term such that $CI^{\bullet}(\ulcorner\psi\urcorner) = \ulcorner\mathcal{CI}(\psi)\urcorner$.

Example 2

The product of polynomials is a polynomial.[2]

Here is its standard mathematical proof: *given a pair of polynomials f, g, using the well-known formula, calculate coefficients of the product polynomial $p_{f\cdot g}$, and prove in arithmetic that*

$$f \cdot g = p_{f\cdot g}. \tag{6}$$

This is a selector proof: for each f, g, it finds a proof of (6) in PA. To formalize this proof in PA, we build

- a p.r. term $Product^{\bullet}(x, y)$ such that $Product^{\bullet}(\ulcorner f \urcorner, \ulcorner g \urcorner)$ is $\ulcorner f \cdot g = p_{f\cdot g} \urcorner$;
- a natural p.r. selector $s(x, y)$ such that $s(\ulcorner f \urcorner, \ulcorner g \urcorner)$ is the Gödel number of a PA-derivation of $f \cdot g = p_{f\cdot g}$.

By direct formalization of the above reasoning, PA proves

$$\forall x, y\ s(x, y) : Product^{\bullet}(x, y).$$

[2] A polynomial is a term $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where each a_i is a numeral and x a variable. In $f \cdot g$, "$\cdot$" stands for the usual PA multiplication.

Example 3

Take the double negation law, DNL[3] in arithmetic: *for any formula* X,

$$X \leftrightarrow \neg\neg X. \tag{7}$$

The standard proof of DNL in PA is *for given* X*, build the usual logical derivation* $D(X)$ *of* (7) *in* PA*.* This is a selector proof which builds an individual PA-derivation for each instance of DNL in a way that provably works for any input X. This proof can be easily formalized in PA as a derivation of

$$\forall x\ s(x) : DNL^{\bullet}(x).$$

Here $DNL^{\bullet}(x)$ is a natural coding term such that

$$DNL^{\bullet}(\ulcorner X \urcorner) = \ulcorner X \leftrightarrow \neg\neg X \urcorner.$$

The selector $s(x)$ is a natural p.r. term such that $s(\ulcorner X \urcorner) = \ulcorner D(X) \urcorner$.

[3] or any other tautology containing propositional variables.

Ubiquitous selector proofs

So, selector proofs have been used as proofs in arithmetic of serial properties

$$\{F(u)\}$$

in which u is a syntactic parameter (ranging over terms, formulas, derivations, etc.) and $F(u)$ is an arithmetical formula for each u. Given u, such a proof selects a PA-derivation for $F(u)$ with subsequent, possibly informal or default, verification of this procedure.

Within this tradition, we have to accept and study selector proofs as legitimate logic objects. In this case, the proof of PA's consistency within PA (below) is a natural consequence.

> *If we don't accept selector proofs, we have to admit that such basic facts as induction, propositional tautologies, product of polynomials, etc., are not provable in arithmetic. Furthermore, consistency is not provable by the same bureaucratic reason: selector proofs are not included.*

Selector proof vs. single formula consistency proofs

Let Con_T be the standard consistency formula for a theory $T \supseteq \mathsf{PA}$. Consider theories:

$$\mathsf{PA}_0 = \mathsf{PA}, \qquad \mathsf{PA}_{i+1} = \mathsf{PA}_i + \mathsf{Con}_{\mathsf{PA}_i}, \qquad \mathsf{PA}^\omega = \bigcup_{i \in \omega} \mathsf{PA}_i.$$

Consider a folklore consistency proof of PA^ω by means of PA^ω:

> *Let D be a derivation in PA^ω. Find n such that D is a derivation in PA_n. $\mathsf{Con}_{\mathsf{PA}_n}$ – one of the postulates of PA^ω – implies that D does not contain $\bot$.*

This is a fine selector proof of consistency of PA^ω in PA^ω. The selector that, given D, computes the code of a PA^ω-derivation of "D does not contain $\bot$" is p.r. and its verification is straightforward.

On the other hand, by G2, PA^ω does not prove $\mathsf{Con}_{\mathsf{PA}^\omega}$ but this observation does not harm the consistency proof above because that proof does not derive $\mathsf{Con}_{\mathsf{PA}^\omega}$.

Selector proofs vs. ω-rule

Question: *What is the difference between selector proofs and ω-rule*

$$\frac{F(0), F(1), \ldots, F(n), \ldots}{\forall x\ F(x)}\ ?$$

(Note that adjoining PA with the ω-rule results in true arithmetic.)

Our response: Selector proofs represent derivations of assumptions $F(0), F(1), \ldots, F(n), \ldots$ in a finite form using verified selector terms:

$$v{:}[\forall x\ s(x){:}F^{\bullet}(x)],$$

but do not make the conclusion $\forall x F(x)$. By doing this, we stay within PA since PA does not necessarily prove $\forall x F(x)$ here.

A basic foundational observation

(i) We provide a selector proof of PA-consistency in $\widehat{\mathsf{PA}}$.

(ii) We then formalize (i) in PA.

One may ask whether we assume the consistency of $\widehat{\mathsf{PA}}$? No, we don't need the consistency assumption to reason within given boundaries.

A sketch of the proof

Step 1 - preliminary. Inspect the well-known proof of reflexivity of PA:

> *Let* $\mathrm{PA}_{\restriction n}$ *be the fragment of* PA *with the first* n *axioms. Then for each numeral* n*,* $\mathrm{PA} \vdash \mathrm{Con}_{\mathrm{PA}_{\restriction n}}$.

From this proof, extract a natural primitive recursive function that given n builds a derivation of $\mathrm{Con}_{\mathrm{PA}_{\restriction n}}$ in PA. Note that $\mathrm{Con}_{\mathrm{PA}_{\restriction n}}$ implies "*D is not a proof of $\bot$*" for any specific derivation D in $\mathrm{PA}_{\restriction n}$.

Step 2 - a contentual selector proof of the consistency of PA. Given a PA-derivation D, find n such that D is a derivation in $\mathrm{PA}_{\restriction n}$ (an easy primitive recursive procedure). By Step 1, PA proves that "*D is not a proof of $\bot$*" and the code of this proof can be calculated by a primitive recursive selector s from the code of D.

Step 3 - internalization. A natural internalization of Step 1 and Step 2 in PA yields the desired verification of the selector in PA:

$$\mathrm{PA} \vdash \forall x\ s(x){:}\neg x{:}\bot.$$

How far we can go with proving consistency in PA

Let $x{:}_T\varphi$ be a shorthand for a proof predicate in a theory T:

"x is a code of a proof of formula φ in T."

We drop this subscript when $T = \mathrm{PA}$.

Suppose PA selector-proves the consistency of a theory T. Then for some primitive recursive term $s(x)$, PA proves

$$\forall x\ s(x){:}\neg x{:}_T\bot.$$

By logical reasoning, PA then would prove

$$\forall x\ \Box\neg x{:}_T\bot$$

which was independently shown to be impossible by Kurahashi and Sinclaire for $T = \mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$. This indicates that PA cannot prove consistency of any theory $T \supseteq \mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$ by the given method without further modifications.

However, PA selector-proves consistency of T for some proper extensions T of PA in particular $T = \mathrm{PA} +$ *slow consistency of* PA (shown by Freund/Pakhomov and Gadsby independently).

What went wrong in the UCT argument in the first place?

By Gödel's second incompleteness theorem, G2, PA, if consistent, does not prove $\mathsf{Con}_{\mathsf{PA}}$. The standard justification of UCT in its Britannica form is that any consistency proof for PA when formalized in PA yields a derivation of $\mathsf{Con}_{\mathsf{PA}}$. Therefore, no consistency proof for PA can be formalized in PA.

A tacit assumption (*) of this argument is that

any formalization of PA*-consistency in* PA *yields* $\mathsf{Con}_{\mathsf{PA}}$ *in* PA.

This assumption (*) is wrong: the natural formalization of the canonical definition of PA-consistency, $\mathsf{Con}^S_{\mathsf{PA}}$, does not imply $\mathsf{Con}_{\mathsf{PA}}$ in PA.

Gödel's Theorem and Hilbert's consistency program

Despite widespread belief, **Gödel's second incompleteness theorem has not canceled Hilbert's consistency program**. In particular, one cannot conclude that Hilbert's ϵ-substitution method for PA, HES, if successful, should *a priori* produce a PA-proof of $\mathsf{Con}_{\mathsf{PA}}$.

> *As we know, HES was not successful in proving of consistency of* PA[4]*. As this is shown above, for some well-principled generalization of HES, selector proofs, the corresponding* PA*-consistency statement is provable in* PA*.*

Since the burden of proof lies on UCT, there were no reasons to assume that G2 has precluded HES from succeeding. Moreover, as UCT is shown to be false, it is time to revive Hilbert's consistency program.

[4]Kripke, in 2022, showed that HES in its original form cannot succeed for non-Gödelian reasons. This, however, does not alter the fact that for 90 years, Hilbert's consistency program remained canceled without sufficient cause.

Some Proof Theory of selector proofs

Let $\varphi(x)$ be an arithmetical formula. By $\{\varphi(x)\}$ we denote a serial property $\{\varphi(0), \varphi(1), \varphi(2), \ldots, \varphi(n), \ldots\}$ which we call *scheme*.

A *proof of a scheme* $\{\varphi(x)\}$ in PA is a pair $\langle s, v\rangle$ where

- s is a primitive recursive term (*selector*),
- v is a PA-proof of $\forall x\, s(x){:}\varphi(x)$ (*verifier*).

The following properties of proofs of schemes hold.

- Proofs of schemes in PA are finite syntactic objects.
- The proof predicate "$\langle s, v\rangle$ is a proof of scheme $\{\varphi\}$" is decidable.
- The set of provable in PA schemes is recursively enumerable.

Gadsby: Iterated selector proofs collapse to non-iterated selector proofs.

Degrees of provability for schemes

A scheme $\{\varphi\}$ is

- *provable in* PA if it has a proof in PA,
- *strongly provable in* PA if PA $\vdash \forall x \varphi(x)$,
- *instance provable in* PA if PA $\vdash \varphi(n)$, for each $n = 0, 1, 2, \ldots$.

Proposition 2.

i) "Strongly provable in PA" yields "provable in PA,"

ii) "provable in PA" yields "instance provable in PA."

Corollary. *Proposition 2(ii)* naturally extends from schemes to all serial properties. Proving serial properties does not add new theorems to PA.

Proposition 3

i) "Instance provable" does not yield "provable,"

ii) "provable" does not yield "strongly provable."

Proof of consistency scheme but not consistency property

Consider the p.r. function $s(x)$ which given x returns a PA-proof of $\neg x{:}\bot$:

> *Given x, check whether x is a proof of $\bot$ in* PA*. If "yes," then put $s(x)$ to be x followed by a simple derivation of $\neg x{:}\bot$ from $\bot$. If "no," then use provable Σ_1-completeness and put $s(x)$ to be a constructible derivation of $\neg x{:}\bot$ in* PA*.*

Let v be an obvious PA-derivation of $\forall x\ s(x){:}\neg x{:}\bot$.

1. Whether $\langle s, v\rangle$ is a proof of the scheme $\{\neg x{:}\bot\}$ in PA?
2. Whether $\langle s, v\rangle$ is a formalized selector proof of PA-consistency?

The answer to (i) is obviously "yes" since $\langle s, v\rangle$ fits the definition.
The answer to (ii) is "no." For the affirmative answer, each $s(n)$, as a contentual argument, should be a $\widehat{\mathsf{PA}}$-proof that n does not contain $\bot$. This condition is not met: $s(n)$ only states that if n contains $\bot$, we could still offer a fake proof of $\neg n{:}\bot$.

Extensions of PA

Consider a theory $T \supseteq$ PA. Does it prove self-consistency? Yes, it does. Indeed, consider a derivation D in T and let $\widetilde{D}$ be the conjunction of the universal closure of all formulas from D. For an appropriate n,

$$\mathrm{PA} \vdash \widetilde{D} \to Tr_n(\widetilde{D}).$$

Here Tr_n is the standard truth formula for all Σ_n arithmetical sentences φ. Note that for all such φ, PA $\vdash \quad Tr_n(\varphi) \ \leftrightarrow \ \varphi$.

Since $T \vdash \widetilde{D}$,

$$T \vdash Tr_n(\widetilde{D}).$$

Since T proves $\neg Tr_n(\bot)$, T proves that $\bot$ in not in D.

This reasoning defines a primitive recursive selector that for each D builds a T-proof of "$\bot$ *in not in* D." The natural internalization of this reasoning in PA (hence in T), confirms that this is a selector proof of T-consistency in T.

Finally: is mathematics able to prove its own consistency?

Selector proofs provide a formal affirmative answer and here is a sketch of the argument.

Note that ZF is *essentially reflexive*. Namely, any theory $T \supseteq \mathrm{ZF}$ proves the standard consistency formula for each of its finite fragments $\{\varphi\}$; given φ we constructively build the proof of $\mathrm{Con}_{\{\varphi\}}$ in T.

This suggests a natural selector proof of consistency of any $T \supseteq \mathrm{ZF}$ in T: given a derivation D on T, calculate the conjunction φ of all formulas in D and find a proof of $\mathrm{Con}_{\{\varphi\}}$ in T. This yields that T proves that D does not contain contradictions. This proof is naturally formalizable in ZF, hence in T.

As before, the mathematical value of such a proof depends on how much trust we invest into T itself.

Any progress in Hilbert Program?

This work dismantles the Unprovability of Consistency thesis in *Encyclopædia Britannica*'s version. It does not intend to reach the principal goals of the Hilbert Program of proving consistency by finitary means. However, it makes progress by removing its major roadblock.

Before: *It was believed that Gödel's second incompleteness theorem prevented the proving of consistency of a system within that system. It was also thought that Hilbert's consistency program could not succeed because of that.*

After: *Gödel's second incompleteness theorem does not apply to Hilbert's approach to proving consistency how it was previously thought. Though the ϵ-substitution method for* PA *fails (by non-Gödelian reasons), some modifications of Hilbertian methods prove consistency of* PA *within* PA.

The results by Gadsby, Kurahashi, and Sinclaire indicate that some new ideas are needed for further progress in Hilbert's consistency program.

Selector proofs beyond proving consistency

Let us revisit the idea of presenting a selector proof of a scheme $\{F(u)\}$ in a simplified implicit form as $\forall x \Box F(x)$, i.e.,

$$\forall x \exists y [y{:}F(x)]. \tag{8}$$

As we noticed earlier (8) is not a substitute of the standard selector proofs in questions of proving consistency of T in T. However, under some additional meta-assumptions about T, i.e., the arithmetic soundness of T, a proof of $\forall x \Box_T F(x)$ in T yields an explicit selector $s(x)$ such that

$$T \vdash \forall x \; s(x){:}F(x).$$

This suggests that, for a sound T in situations where a specific selector is not essential, one can use (8) in place of the standard selector provability. Studying selector provability in the (8) form becomes an integral part of the selector proof theory, cf. the talk *Properties of Selector Proofs* by E. Gadsby at this conference.

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